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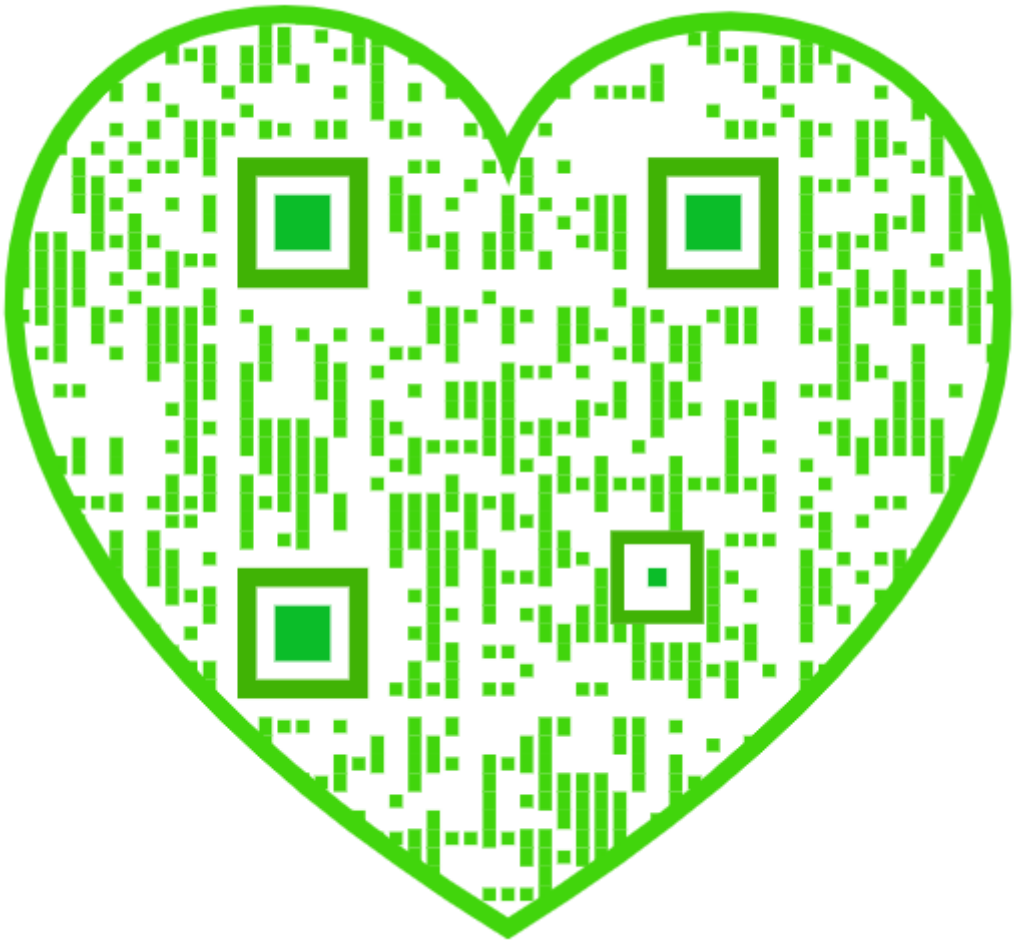

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NOTES ON HYDRAULICS.

PREPARED FOR THE USE OF THE STUDENTS

OF THE

CIVIL ENGINEERING DEPARTMENT

OF THE

MASS. INSTITUTE OF TECHNOLOGY,

BOSTON, MASS.

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W. J. SCHOFIELD, PRINTER, 105 SUMMER STREET.
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6-13906

NOTES ON HYDRAULICS.

"*A perfect fluid* is an aggregation of particles which yields at once to the slightest effort made to separate them from each other." A perfect fluid has no cohesion, offers no resistance to change of shape, assumes the shape of the vessel containing it, and its shape may be changed without doing any internal work.

Fluids are divided into liquids and gases: the former are incompressible and inelastic; the latter are compressible, elastic, tend to expand indefinitely, and therefore vary in density.

No known fluid is perfect, but all offer some resistance to change of shape. An imperfect fluid, however, yields to the slightest effort made to separate its particles from each other, if that effort be continued long enough.

The mechanics of fluids is divided into Hydrostatics, Hydrodynamics, Aerostatics, and Aerodynamics. The general term hydraulics may be held to include them all, though generally limited to the first two.

VARIATION IN DENSITY OF GASES.—Let v be the volume, w the weight, p the pressure, and t the temperature of a given quantity of a gas. Then γ, the weight of a unit of volume, is equal to $\frac{w}{v}$. The density ρ may be defined as the mass of a unit of volume, hence it is equal to $\frac{\gamma}{g}$, or $\rho\, g = \gamma = \frac{w}{v}$; $\rho\, v =$ constant.

Boyle's or Mariotte's law.—With a constant temperature, the pressure varies inversely as the volume; hence $p\, v =$ constant, and $p = c\, \rho$, where c is a constant.

Gay-Lussac's law. —With a constant pressure, an increase of temperature of 1° C. produces in a mass of air an expansion of 0.003665 of its volume at 0° C.

Letting subscript $_o$ refer to a temperature of 0° C., we have for the volume at any temperature t

$$v = v_o\ (1 \times a\ t) \qquad \text{hence}$$
$$\rho_o = \rho\ (1 + a\ t).$$

At a constant temperature 0° C., pressure and density vary according to the law $p = c\,\rho_o$, hence substituting for ρ_o we have

$$p = c\rho\ (1 + a\ t) = \frac{c}{g}.\ \gamma\ (1 + a\ t)$$
$$= k\,.\,\gamma\ (1 + a\ t)$$

If γ_1 be the weight of a cubic unit under a pressure p_1 and temperature t_1, and γ_2 correspond to p_2 and t_2, we have

$$p_2 = k\,\gamma_2\ (1 + a\ t_2)$$
$$p_1 = k\,\gamma_1\ (1 + a\ t_1) \qquad \text{hence}$$
$$\frac{\gamma_2}{\gamma_1} = \frac{p_2}{p_1} \cdot \frac{1 + a\ t_1}{1 + a\ t_2}$$

CHAPTER I.

HYDROSTATICS.

Hydrostatics treats of liquids at rest; its theorems apply to viscous as well as perfect liquids, for the pressures in both follow the same laws. It is only when motion is considered that viscosity has any effect.

I. *In a perfect fluid always, and in all fluids at rest, the pressure on any plane is normal to that plane;* for otherwise there would be a tangential component which would cause motion. The *intensity* of pressure is the pressure per square unit of surface.

II. *At any point in any fluid at rest, or in a perfect fluid in motion, the pressure has the same intensity on all planes passing through that point.*

PROOF (Fig. 1).—Let O X, O Y, and O Z be three rectangular axes, and O A B C any triangular pyramid. The pressures p_x p_y p_z, and p per square unit on the planes O C B, O A C, O A B, and A B C, are normal to those planes. The total pressures are:

p_x · O B C parallel to X.
p_y · O A C parallel to Y.
p_z · O A B parallel to Z.
p · A B C normal to A B C.

The forces acting on the pyramid must be in equilibrium; these forces are the four just named, and the external forces (such as gravity) acting on the mass O A B C, the latter being proportional to the volume of the pyramid. Let the pyramid be made smaller, by moving the plane A B C parallel to itself, towards O. The four pressures, since they vary as the areas of the sides, vary as the square of any dimension, such as O A. The internal forces acting on the mass O A B C, since they vary as the volume, vary as the cube of any dimension, such as O A. Hence at the limit, when the pyramid becomes infinitesimal, the latter will vanish in comparison with the four pressures, and these pressures alone must be in equilibrium. Resolving them parallel to the three axes, and putting first the components parallel to X equal to zero, we obtain the following: calling α, β, and γ the angles which p makes with O X, O Y, and O Z, the component of the pressure p . A B C parallel to O X is p . A B C. cos. α. But A B C. cos. α = B O C, hence p. A B C. cos. $\alpha = p$. B O C. The condition that the components parallel to O X are in equilibrium becomes therefore

$$p_x \cdot \text{B O C} = p \cdot \text{B O C} \quad \therefore$$
$$p_x = p \cdot$$

In a similar way we should find $p_y = p$ and $p_z = p$, hence we have

$$p_x = p_y = p_z = p \cdot$$

which proves the proposition, as the result is independent of the particular plane A B C.

The following problem is one of the principal general problems in hydrostatics. We give first the general solution; particular cases may, as we shall see, be solved more easily.

Problem. – To find the differential equation of pressure in any liquid.

We consider simply the equilibrium of an infinitely small parallelepipedon A B C D E F G H (Fig. 2), whose edges are parallel to three rectangular axes. The weight of a unit of volume is γ, and the external force on a unit of mass is X along the axis of X, Y along the axis of Y, and Z along the axis of Z. The pressure at A per unit of area is p, and the edges A B, A C, and A E are respectively dx, dy, and dz. The inten-

sity of pressure on the plane A C G E, whose area is $dy\,dz$, is p. That on the plane B F H D, which has the same area, but is farther from O by a distance dx, will not be exactly the same, but will be $\left(p + \frac{\delta p}{\delta x}.\ dx\right)$. The only forces acting on the parallelpipedon parallel to O X are these two pressures, acting in opposite directions, and the force X per unit of mass. Placing these equal to zero, since they must be for equilibrium, we have

$$0 = p\,dy\,dz - \left(p + \frac{\delta p}{\delta x}dx\right) dy\,dz + X.\ \gamma.\ \frac{dx.\ dy\,dz;}{g}$$

or, reducing,

$$\frac{\delta p}{\delta x} = \frac{\gamma}{g}.\ X$$

This expresses the rate at which the pressure p changes as x is increased.

In a similar way we should have

$$\frac{\delta p}{\delta y} = \frac{\gamma}{g}.\ Y$$

$$\frac{\delta p}{\delta z} = \frac{\gamma}{g}.\ Z$$

If we change x, y, and z simultaneously, for instance, if we go from A to H in the figure, we may pursue the path A B D H, and the total change in p will be

$$\frac{\delta p}{\delta x}.\ \text{A B} + \frac{\delta p}{\delta y}.\ \text{A D} + \frac{\delta p}{\delta z}.\ \text{D H}, \quad \text{or}$$

$$dp = \frac{\delta p}{\delta x}.\ dx + \frac{\delta p}{\delta y}.\ dy + \frac{\delta p}{\delta z}.\ dz$$

$$= \frac{\gamma}{g}\left[X\,dx + Y\,dy + Z\,dz\right]$$

and if p is the pressure at A, then $p + dp$ will be the pressure at H. This is then the general differential equation of pressure.

For surfaces of equal pressure $dp = 0$, or $X\,dx + Y\,dy + Z\,dz = 0$. The free surface is a surface of equal pressure, and satisfies this equation.

Application to liquids under action of gravity alone. (Fig 3.)

Here $X = Y = 0$; $Z = -g$. Hence equation of surfaces of equal pressure is

$$o = -g\,dz, \qquad \text{whence}$$

$$z = \text{constant (horiz. planes)}.$$

To find pressure at any point

$$dp = -\gamma\,dz.$$

Let there be a pressure p_o on the surface, then integrating we get

$$p = p_o - \gamma z$$

if our origin is at the surface, and z positive upwards. If z is positive downwards,

$$p = p_o + \gamma z.$$

This may be more easily obtained directly, by considering a vertical cylinder of area unity. Thus the pressure on the bottom is evidently $p_o + \gamma z$, as before. (Fig. 3.)

Application to gases.—In this case γ varies, and if γ_o be the weight of a cubic unit at 0° C. and a pressure p_o, we shall have

$$\gamma = \gamma_o \frac{p}{p_o (1 + a t)} \qquad \text{hence}$$

$$dp = -\gamma_o \cdot \frac{p}{p_o (1 + a t)} \, dz.$$

$$z_1 - z = \log_e \cdot \frac{p}{p_1} \cdot \frac{p_o (1 + a t)}{\gamma_o}.$$

$z_1 - z$ is the difference of heights corresponding to barometric pressures p_1 and p. Corrections: (1) for latitude; (2) for elevation; (3) for temperature of mercury; (4) for temperature of air.

PASCAL'S THEOREM. — Any pressure on a liquid is transmitted with equal intensity throughout the liquid. (Fig. 4.)

Proof. — Take any cylinder whose area is constant, but whose axis has any shape (see fig. 4). Then by the principle of virtual velocities, since the forces along the circumference of the cylinder act normal to its axis, we have

$$p \text{ A. } dx = p' \text{ A } dx \quad \therefore \quad p = p'.$$

This considers effect of pressure alone, and neglects gravity. (Examples of effect of both.) Applications to hydraulic press.

PRESSURES IN LIQUIDS. —We have seen that at any depth d,

$$p = p_o + \gamma d. \quad \text{or,} \quad \frac{p}{\gamma} = \frac{p_o}{\gamma} + d.$$

$\frac{p}{\gamma}$ is called the *height due to the pressure.*

(Graphical representation, Fig. 5.)

Case of liquids of different densities (Fig. 6).—Surfaces of separation are horizontal.

Then $p_1 = p_o + \gamma_1 d_1$

$$p_2 = p_1 + \gamma_2 d_2 = p_o + \gamma_1 d_1 + \gamma_2 d_2.$$

$$p_3 = p_2 + \gamma_3 d_3 = p_o + \gamma_1 d_1 + \gamma_2 d_2 + \gamma_3 d_3, \text{ etc.}$$

APPLICATION OF THE GENERAL EQUATION OF PRESSURE TO A ROTATING LIQUID.—Let us endeavor to find the form of the free surface of a liquid rotating about a vertical axis. Let A B C D (Fig. 7) be a cylindrical vessel containing the liquid, which is rotated about the axis O Z with an angular velocity w. If we suppose the co-ordinate system O X Y Z to revolve, then the liquid is at rest with reference to the system, and we can apply the equations of hydrostatics, provided we take account of the fact that we have a case of relative rest only. Now, we know from mechanics that if a body is actually in motion, and we wish to investigate its motion with relation to a co-ordinate system which is itself in motion, we must add to or compound with the actual velocity of the body a velocity equal and opposite to the velocity which the point occupied by the body would have as a part of the moving co-ordinate system. In the same way, if we are considering forces, then, to refer the case to a moving system, we must compound with the actual forces acting on the body a force equal and opposite to the force which would give the body the motion which the point it occupies has as a part of the moving system. Now, in this case the liquid is actually in motion; but at rest relative to a system revolving with an angular velocity w. If m is the mass of a particle at a distance r from O Z, then the actual force acting on m is its weight or $w\,g$; the force which would give that mass a motion of rotation about O Z with an angular velocity w is $m\,w^2\,r$ acting *towards* O Z; hence, if we wish to refer the motion of the revolving body to the revolving axis, we must compound with its weight a force $m\,w^2\,r$ acting *from* O Z. As the body is at rest with reference to the rotating system, then with these forces the equations of hydrostatics apply. Resolving the centrifugal force $m\,w^2\,r$ parallel to X and Y we shall have on the unit of mass

$$X = w^2\,x \;;\; Y = w^2\,y \;;\; Z = -g. \qquad \text{hence}$$

$$d\,p = \frac{\gamma}{g}\left[w^2\,x\,d\,x + w^2\,y\,d\,y - g\,d\,z\right]$$

and the equation of a surface of equal pressure is

$$g\,d\,z = w^2\left(x\,d\,x + y\,d\,y\right) \quad \text{or}$$

$$z = \frac{w^2}{2g}\cdot\left(x^2 + y^2\right) + \mathrm{C}$$

$$= \frac{w^2}{2g}\cdot r^2 + \mathrm{C}.$$

Hence, in any vertical section of the liquid the free surface and the surfaces of equal pressure are parabolas with a vertical axis, or the free surface is a vertical paraboloid of revolution.

Geometrical proof. — Let P S be $m\,w^2\,r$ and P U $= m\,g$; then P T is resultant, and must be normal to free surface if S is a particle at the free surface. Produce P T to meet O Z at N. Then

$$\text{P M} : \text{N M} : : m w^2 r : m g : : w^2 r : g.$$

hence $\text{N M} = \frac{g}{w^2} = \text{constant}$; or the subnormal of the curve is constant, a property only of the parabola.

Application to a mass revolving about a horizontal axis (Fig. 8). — In this case, which represents a vertical water-wheel with buckets,

$$X = w^2 x\ ;\ Y = 0;\ Z = w^2 z - g,$$

$$\text{hence}\quad d p = \frac{\gamma}{g}\left[w^2 x\,d x + w^2 z\,d z - g\,d z\right]$$

Equation of surface of equal pressure is

$$\frac{w^2}{z}\left(x^2 + z^2\right) - g z + \text{C}^2\,0, \qquad \text{or}$$

$$x^2 + z^2 - z\,\frac{2g}{w^2} = \text{C}'. \qquad \text{or}$$

$$x_2 + \left(z - \frac{g}{w^2}\right)^2 = a^2 \qquad \text{if we call}$$

$$\text{C}' + \frac{g^2}{w^4}. = a^2.$$

This is the equation of a circle whose center is at M, O M being $\frac{g}{w^2}$. Hence the free surface in all the buckets are parts of circles whose common centre is at M.

Geometrical proof. — Let P S represent the centrifugal force, and P G the weight. Then P T is resultant, and is normal to free surface. Produce P T to M. Then

$$\text{O M} : \text{O P} :: m g : m w^2 r :: g : w^2 r$$

$$\text{O M} = \frac{r g}{w^2 r} = \frac{g}{w^2}.$$

This theorem is applied in the study of water-wheels, as the shape of the free surface evidently determines the point at which the buckets will begin to empty, as the wheel revolves.

Liquid moving in a straight line (Fig. 9). — If a liquid is moving horizontally in a straight line with a constant velocity v, then the only force acting on it is its weight, and the force necessary to maintain it at the velocity v is zero, because that velocity is constant. Hence the free surface will be horizontal, just as though the liquid were at rest. But if it be moving with an *acceleration* a, then the liquid is at rest with reference to a system moving with the same acceleration, and in order to apply the equations of hydrostatics we must add to its weight a force $m\,a$ in a direction opposed to the motion. Let the motion be at an angle α with the horizontal.

$$X = -\,a\cos.\,\alpha\;;\quad Y = 0\;;\quad Z = -\,g - a\sin.\,\alpha.$$

$$dp = \frac{\gamma}{g}\left[-\,a\cos.\,\alpha\,dx - a\sin.\,\alpha\,dz - g\,dz\right]$$

Equation of surface of equal pressure is

$$-\,ax\cos.\,\alpha - az\sin.\,\alpha - gz = \mathrm{C}.$$

$$z = -\frac{a\cos.\,\alpha}{a\sin.\,\alpha + g}\,x + \mathrm{C},\quad \text{which represents a straight}$$

line inclined at an angle ϕ with the horizontal, and

$$\tan.\,\phi = -\frac{a\cos.\,\alpha}{a\sin.\,\alpha + g}$$

Pressure on vertical or inclined surfaces (Fig. 10). — We shall now consider the pressure on plane surfaces immersed in liquids. Let A B be an immersed plane, making an angle α with the surface of the liquid O O′. If there is liquid on both sides of A B, then the pressure on each side is the same, and the resultant pressure zero. Our problem is to find the pressure on one side. If A B is the side of the vessel containing the liquid, there is the atmospheric pressure outside, partly counteracting the pressure from within. The atmospheric pressure p_o on the free surface O O′, however, is transmitted uniformly throughout the liquid, just balancing that outside, and we shall leave it out of consideration. Let us divide the area A B into a series of horizontal strips; let b be the length and dx the breadth of each strip. Then the pressure on the strip, if h is its depth below the surface is $b\,dx.\,\gamma\,h$, or $b\,dx.\,\gamma.\,x\sin.\,\alpha$, x being the distance from O. Hence the total pressure on the plane is.

$$\mathrm{P} = \int b\,dx.\,\gamma\sin.\,\alpha.\,x$$

$$= \gamma\sin.\,\alpha\int_{x_o}^{x_1} b\,x\,dx.$$

The quantity under the integral is the moment of the area about O, hence equal to A times the distance of centre of gravity x_g, hence

$$P = \gamma \sin. a \, . \, A \, . \, x_g = A \, \gamma . \, x_g \sin. a$$

$= A \, . \, \gamma \, h_g$; if h_g is the depth of the centre of gravity.

Hence the theorem : —

The total pressure on a plane surface immersed is equal to the area multiplied by the pressure at the centre of gravity of the surface; or the mean pressure is the pressure at the centre of gravity. The pressure will always be the same if the centre of gravity is at the same depth.

Graphical representation of pressure (see fig. 11).

The pressure on the plane increases in intensity from A to B. The *centre of pressure* is the point of application of the resultant pressure; it is found as follows: the total pressure is the resultant of a series of parallel forces, and we get its point of application by taking moments about two rectangular axes, one passing through O perpendicular to the paper, and one at right angles to this axis and lying in the plane of the surface. Generally, however, all we care for is the depth of the centre of pressure, or its distance from O, hence we need only take moments about the first axis. The moment of the pressure on the strip $b \, d \, x$ is

$$b \, d \, x \, . \, \gamma \, x \sin. a \, . \, x \quad ; \quad \text{hence}$$

$$x_c = \frac{\int_{x_o}^{x_1} b \, d \, x \, . \, \gamma \, x^2 \sin. a}{P} \quad \text{or}$$

$$x_c = \frac{\gamma \sin. a \int_{x_o}^{x_1} b \, x^2 \, d \, r}{\gamma \sin. a \int_{x_o}^{x_1} b \, x \, d \, x}$$

In using these equations, b is of course a variable, and must first be found in terms of x, after which the integration may be effected. The centre of pressure is always below the centre of gravity.

Examples. — 1. Rectangle. $b =$ constant.

$$P = b \, \gamma \sin. a \, . \, \frac{x_1^2 - x_o^2}{2} .$$

$$x_c = \tfrac{2}{3} \, \frac{x_1^3 - x_o^3}{x_1^2 - x_o^2} .$$

If $x_o = o$,

$$P = b \, \gamma \sin. a \, . \, \frac{x_1^2}{2} .$$

$$x_c = \tfrac{2}{3} \, x_1 .$$

2. Triangle with base horizontal and vertex upward.

$$b = b_1 \frac{x - x_o}{x_1 - x_o}. \qquad (b_1 = \text{base.})$$

$$P = \frac{b_1 \gamma \sin. a}{x_1 - x_o}\left[\frac{x_1^3 - x_o^3}{3} - x_o \frac{x_1^2 - x_o^2}{2}\right]$$

$$x_c = \frac{\frac{x_1^4 - x_o^4}{4} - x_o \frac{x_1^3 - x_o^3}{3}}{\frac{x_1^3 - x_o^3}{3} - x_o \frac{x_1^2 - x_o^2}{2}}$$

If $x_o = o$;

$$P = b_1 \gamma \sin. a \cdot \frac{x_1^2}{3}$$

$$x_c = \tfrac{3}{4} x_1$$

3. Triangle with base horizontal and vertex down.

$$b = b_o \frac{x_1 - x}{x_1 - x_o} \qquad (b_o = \text{base.})$$

$$P = \frac{b_o \gamma \sin. a}{x_1 - x_o}\left[x_1 \cdot \frac{x_1^2 - x_o^2}{2} - \frac{x_1^3 - x_o^3}{3}\right]$$

$$x_c = \frac{x_1 \cdot \frac{x_1^3 - x_o^3}{3} - \frac{x_1^4 - x_o^4}{4}}{x_1 \frac{x_1^2 - x_o^2}{2} - \frac{x_1^3 - x_o^3}{3}}$$

If $x_o = o$;

$$P = \frac{b_o \gamma \sin. a}{6} \cdot x_1^2$$

$$x_c = \frac{x_1}{2}.$$

4°. Trapezoid. Bases horizontal. Upper base b_o, lower base b_1.

$$b = b_o + (b_1 - b_o) \frac{x - x_o}{x_1 - x_o}.$$

$$P = \gamma \sin. a \left[b_o \cdot \frac{x_1^2 - x_o^2}{2} + \frac{b_1 - b_o}{x_1 - x_o}\left(\frac{x_1^3 - x_o^3}{3} - x_o \frac{x_1^2 - x_o^2}{2}\right)\right]$$

$$x_c = \frac{b_o \cdot \frac{x_1^3 - x_o^3}{3} + \frac{b_1 - b_o}{x_1 - x_o}\left[\frac{x_1^4 - x_o^4}{4} - x_o \frac{x_1^3 - x_o^3}{3}\right]}{b_o \cdot \frac{x_1^2 - x_o^2}{2} + \frac{b_1 - b_o}{x_1 - x_o}\left[\frac{x_1^3 - x_o^3}{3} - x_o \frac{x_1^2 - x_o^2}{2}\right]}$$

PRESSURE ON CURVED SURFACES. — Here the pressure varies in direction, being everywhere normal to the surface. To find the total pressure the surface must be divided into small areas, which may be assumed as plane, and the pressures on these must be found, and also the centres of pressure. Combining the separate pressures, we get the total pressure and its point or line of application.

PRESSURE ON BOTH SIDES OF A SURFACE. — In Fig. 12, pressure on the left is at any depth h, $p_1 = \gamma h + p_o$, and on the right $p_2 = \gamma h_1 + p_o$; hence the resultant pressure is $p = \gamma (h - h_1) = \gamma H$. Hence there is a constant intensity of resultant pressure below the lower water surface.

The total pressure per foot on A B is

$$H_1 \cdot \gamma H + \gamma \frac{H^2}{2}.$$

The depth of the centre of pressure below A is

$$x_c = \frac{H_1 \left(H + \frac{H_1}{2} \right) + \frac{1}{3} H^2}{H_1 + \frac{H}{2}}$$

PRESSURE IN A PARTICULAR DIRECTION. —Consider, in fig. 1, an infinitely small plane area A B C, subjected to a pressure whose intensity is p. Then the total pressure on A B C is p. A B C. The component of that pressure parallel to X is p. A B C. cos. a, or p. A B C. In a similar way the component parallel to Y is p. O A C, and parallel to z is p. O A B. Hence we have the theorem : —

The component parallel to any given direction, of the pressure on an element of area, is equal to the intensity of pressure multiplied by the projection of that element of area on a plane at right angles to the given direction.

If we consider a finite plane surface, then the pressures on each element are parallel; if A be any element, then its projection will be n A, in any given direction, and if h be the height due to the pressure on the element, then the total pressure in the given direction will be $\Sigma \gamma h . n$ A. In this case, the following rule evidently holds good; *the pressure in any particular direction against a plane area is equal to the weight of a column of water whose base is the projection of the given surface on a plane perpendicular to the given direction, and whose height is the depth below the surface, of the centre of gravity of the given area.* For, in this case, the total pressure is $n \Sigma \gamma h$. A. For curved surfaces, however, the law does not, in general, hold; for the ratio of any element to its projection will not be constant, and we shall have for the total pressure $\Sigma \gamma h . n$ A. in which n is a variable. For some curved surfaces — as, for instance, a cone the ratio n

will be constant for some planes, as one parallel to the axis, in which case the same law holds as for plane surfaces.

Generally we wish to find vertical or horizontal pressures; for the former we therefore take the horizontal projection, and, for the latter, the vertical projection of the given surface. (Examples: horizontal pressure on inclined reservoir wall; pressure in pipes and spheres.) The following rule, however, may be stated in regard to horizontal pressures: *the horizontal pressure of a liquid against any surface whatever, plane or curved, equals the weight of a column of water whose base is the area of the vertical projection of the given area, and whose depth is the depth of the centre of gravity of the projected area.*

Equilibrium of immersed bodies. — Let fig. 13 represent a body immersed in water. On any element of its surface, as $a\ b$, there acts a pressure $\gamma\ h$. A, if A be the area. This pressure acts normal to the surface, and may be resolved into three rectangular components, V, H, and H_1, the latter acting perpendicular to the paper. Let $a\ b\ c\ d$ be a horizontal cylinder in the direction of H, and $a\ b\ c'\ d'$ a vertical cylinder in the direction of V. Then $H = \gamma\ h.\ F'$ and $V = \gamma\ h.\ F$, if F' and F are the areas of these cylinders. Now, it is clear that H will be balanced by an equal force on on $c\ d$, and in the same way H_1 would be balanced, so that clearly *the horizontal pressures on any immersed body are exactly balanced.* With regard to the vertical pressures, the cylinder $a\ b\ c'\ d'$ is acted upon by its weight and the two pressures V and V_1, the resultant of which is $V - V_1 = V_o = \gamma\ (h - h_1)\ F$ = weight of a volume of liquid equal to the volume of the cylinder $a\ b\ c'\ d'$. Hence the resultant pressure exercised on any cylinder as $a\ b\ c'\ d'$ is an upward pressure equal to the weight of its volume of liquid. Hence it follows that, *any body immersed in a liquid, wholly or partially, loses a weight equal to the weight of the liquid displaced.* There are two forces thus acting on such a body, namely, its weight and the upward pressure of the water. The former acts through the centre of gravity of the body. Regarding the latter, the upward pressure on any cylinder as $a\ b\ c'\ d'$ is proportional to the volume of the cylinder, regardless of its weight; hence the resultant upward pressure on the entire body acts through the centre of gravity of the *volume of water displaced.* If the body were homogeneous, and entirely immersed, the centre of gravity of the body would coincide with that of the volume displaced, but not if the body were unhomogeneous.

In order that an immersed body may be in equilibrium, its centre of gravity and that of the displaced water must lie in the same vertical, or else a moment must be applied to the body to maintain its equilibrium.

If a body weighs more than its volume of liquid, it will tend to sink indefinitely. (Specific gravity greater than unity.) If it weighs less than its volume of liquid, it will tend to rise above the surface, until its own weight is exactly

equal to the weight of the displaced liquid. This enables us to find how deep a body, whose specific gravity is less than unity, will sink.

Let fig. 14 represent an immersed body. Let W be its weight, acting through its centre of gravity G, and P the upward pressure of the liquid, acting through the centre of gravity of volume C. If the line G C is vertical, the body does not tend to rotate (lower fig). If G is below C, then the body is in stable equilibrium, for if turned, as in the upper figure, the couple formed will tend to bring it back to its former position. If G is above C, then the equilibrium is unstable. C and G are fixed points in the body, and C is called the *metacentre;* hence, for stability, the metacentre should be above the centre of gravity. If the two coincide, the body is in equilibrium in any position.

The same theorems are true of bodies whose specific gravity is less than unity, and which float upon the surface. (Applications: raising bodies out of water; drawing out piles; lessening draught of ships, etc.)

Stability of floating bodies. — In bodies which are entirely submerged, the centre of gravity of the body, and also that of the displaced water, are fixed points. Hence the two forces, weight and buoyancy, always act through fixed and definite points. With a floating body the case is different, and the problem much more difficult. (See figs. 15 and 16.)

Let C and S be the centres of gravity of the body, and of the displaced liquid, when K F is vertical. Now, if the body be rotated, as in fig. 15, the point S is transferred to S_1, towards the side of greater immersion, and the weight P acting downward through C, with the buoyancy, equal to P, acting upward through S_1, form a couple tending to rotate the body. If M is the point where the upward force cuts the axis F K, then the equilibrium is stable if M is above C, and unstable if M is below C, and the moment of stability is $P \cdot C D$, or $P\, c \sin. \phi$, if c is the distance C M, and ϕ the angle of rotation.

This case is generally treated, as first proposed by Bouguer (1746), under the supposition that the area H K R remains constant. This might be so with bodies of the same cross-section throughout, but with ordinary bodies, and ships, whose cross-section varies, the immersed area at any one section need not remain constant. And even with bodies of constant section it should be supposed that they might have their equilibrium disturbed in such a way as for a moment to displace a greater or less quantity of water, thus changing the area immersed, in which case the body would be thrown into vibration. A more general solution than Bouguer's has been given by Duhamel, the results of which are the following: —

Let G be the centre of gravity of the body, and O that of the water displaced; call the section of the body by the water-surface the plane or area of floatation; let I be the moment of inertia of this area about an

axis passing through its centre of gravity, and lying in its plane; and let V be the volume of the liquid displaced. Then the following theorem is true: —

To ensure stable equilibrium, it is necessary and sufficient that the centre of gravity of the body, G, *should be below a point* P, *which point* P *is located at a distance* O P *above the centre of gravity of the water displaced*, O, *and in the line* O G, *equal to* $\frac{I}{V}$. *The point* P *is called the metacentre.*

Hence, in this case, the point G need not be below O, but only below P. The value of I will vary with the axis about which it is taken, hence we must take its least value.

The general problem of the stability and oscillations of a body exposed to waves is yet far from being solved.

(Applications in ship-building. For approximate treatment, see Weisbach's *Mechanics*.)

Example. — In fig. 17 we have $I = \frac{1}{12} b\, l^3$ (or $l\, b^3$, according as l or b is the smaller); $V = b\, h\, l.\, s$ if $s =$ sp. gravity of the body; $\therefore$ equilibrium is assured if centre of gravity is below P, O P being $\frac{l^2}{12\, hs}$. We have O G $= \frac{h - y}{2} = \frac{h\,(1 - s)}{2}$. Hence equilibrium is stable if

$$\frac{h\,(1 - s)}{2} < \frac{l^2}{12\, h\, s} \quad \text{or if}$$

$$\frac{l}{h} \quad \text{or} \quad \frac{\text{smallest horizontal dimension}}{h} > \sqrt{6\, s\, (1 - s)}$$

For $s = \frac{1}{2}$ we have $\frac{l}{h} > 1.225$; hence smallest horizontal dimension must be greater than $1.225\, h$.

Specific gravity. — Let V = volume of a body, s_1 its specific gravity, s_2 the specific gravity of liquid in which it is immersed. Then its weight is

$$G = V\, \gamma\, s_1$$

and the weight it loses when immersed is

$$F = V\, \gamma\, s_2\ ; \qquad \text{hence}$$

$$\frac{G}{F} = \frac{s_1}{s_2}\ ;\ s_1 = s_2 \frac{G}{F}.$$

If we immerse in water, $s_1 = \frac{G}{F}$.

If s_1 is less than unity, we may attach the body to a heavier body whose specific gravity is known. Let

G_1 = weight of heavy body; V_1 its volume; s_1 its specific gravity.
G_2 = weight of light body; V_2 its volume; s_2 its specific gravity.
S = specific gravity of combination.

Then $G_1 = \gamma s_1 V_1$; $G_2 = \gamma s_2 V_2$.

$$G_1 + G_2 = \gamma s (V_1 + V_2) ;$$

but $$V_1 + V_2 = \frac{G_1}{\gamma s_1} + \frac{G_2}{\gamma s_2}$$ hence

$$G_1 + G_2 = \gamma s \left(\frac{G_1}{\gamma s_1} + \frac{G_2}{\gamma s_2}\right)$$

From this we find

$$s_2 = \frac{G_2}{\frac{G_1}{s} + \frac{G_2}{s} - \frac{G_1}{s_1}}$$

These principles enable us to determine the specific gravity of any solid substance.

Liquids. — Let G = weight of an empty vessel; G_1 its weight when filled with water; G_2 when filled with the given liquid; then

$$s = \frac{G_2 - G}{G_1 - G} .$$

(Specific gravity of liquids also easily found by hydrometers. See Physics.)

HYDRODYNAMICS.

Before proceeding to the subject of hydrodynamics, it will be convenient to recall some of the principles of mechanics which we shall most apply.

1.° The force F, which imparts in a unit of time an accleration a to a body of mass m, is equal to $m\,a$.

2.° The work which a force F does in moving its point of application over a distance $d\,s$, making an angle α with the direction of F, is $F\,d\,s\,.\,\cos.\,\alpha$.

3.° To change the velocity of a body from v to v_1 requires a work $W = \frac{1}{2}\,m\,(v_1^2 - v^2.)$

4.° If v is the velocity, a the acceleration, and F the force acting on a body of mass m, and if these be resolved along three rectangular axes X, Y, and Z, and the components denoted by subscript letters, then we shall have

$$F_x = m\,a_x \;;\; F_y = m\,a_y \;;\; F_z = m\,a_z\,.$$

$$v_x = v\cos.\,a \;;\; v_y = v\cos.\,\beta \;;\; v_z = v\cos.\,\gamma\,.$$

if a, β, and γ are the angles which v makes with the axes of X, Y, Z respectively.

From these we get

$$F_x\,d\,t = m\,a_x\,d\,t = m\,d\,v_x$$

$$F_y\,d\,t = m\,a_y\,d\,t = m\,d\,v_y$$

$$F_z\,d\,t = m\,a_z\,d\,t = m\,d\,v_z\,.$$

and if the velocities are changed from v_x^o, v_y^o, v_z^o at the time t_o to v_x', v_y', v_z', at the time t_1, then we shall have

$$m\,(v_x' - v_x^o) = \int_{t_o}^{t_1} F_x\,d\,t\,. \qquad \text{etc.}$$

In words, we may say that *mass multiplied by change in velocity equals force multiplied by time.*

From the equation

$$F_x\,d\,t = m\,d\,v_x \qquad \text{we get}$$

$$F_x\,v_x\,d\,t = m\,.\,v_x\,d\,v_x = F_x\,d\,x\,.$$

Hence

$$F_x\,d\,x = m\,v_x\,d\,v_x$$

$$F_y\,d\,y = m\,v_y\,a\,v_y$$

$$F_z\,d\,z = m\,v_z\,d\,v_z\,.$$

and integrating the first of these we get

$$\tfrac{1}{2}\,m\,(v_x'^2 - v_x^{o2}) = \int_{t_o}^{t_1} F_x\,d\,x$$

or, in words: *force multiplied by space equals change in energy.*

These equations apply to a material point. If we wish to apply them to a finite body of mass M, we must take account not only of the external forces but of the internal forces. Thus if m denote the mass of an elementary particle, we shall have, summing up for all these particles

$$\Sigma\,m\,v_x' - \Sigma\,m\,v_x^o = \Sigma\int_{t_o}^{t_1} F_x\,d\,t\,. + \Sigma\int_{t_o}^{t_1} f_x'\,d\,t$$

$$+ \Sigma\int_{t_o}^{t_1} f_x''\,d\,t \qquad \text{etc.}$$

in which f', f'', are the interior forces exerted on the particle by the other particles of the system. But as action and reaction are equal and opposite, if we extend our summation over all the particles of the body, these terms with f', etc., will each be balanced by a corresponding term with opposite sign. Hence we shall have

$$\Sigma\, m\, v'_x - \Sigma\, m\, v^o_x = \Sigma \int_{t_o}^{t_1} \mathrm{F}_x\, d\, t\,.$$

Regarding the change of energy, the work done by the inner forces will not reduce to zero, but we shall have

$$\Sigma\, \tfrac{1}{2}\, m\, v_x'^2 - \Sigma\, \tfrac{1}{2}\, m\, v_x^{o\,2} = \Sigma \int_{t_o}^{t_1} \mathrm{F}_x\, d\, x + \Sigma \int f_x'\, d\, x + \Sigma \int f_x''\, d\, x \quad \text{etc.}$$

With a fluid which is perfect and incompressible, the inner forces can do no work, and the terms with f', etc., vanish.

Permanent motion. — *Permanent, as distinguished from variable motion, occurs when at any time, and through the entire volume of the fluid, the particles which pass any particular point have the same velocity (both in amount and in direction), the same pressure, and the same density.* These quantities vary from point to point of the fluid, but are constantly the same at any one point. It is with permanent motion that we have principally to deal.

Bernouilli's theorem (fig. 18). — This theorem is one of the most important in the whole subject of hydraulics, and was first demonstrated by Daniel Bernouilli, in 1738, in his *Hydrodynamica*. We suppose the motion permanent, and the fluid perfect. Let, now, A B be the path of a particle, or, rather, a cylinder inclosing the path of a number of particles. Its normal section is very small, and is w at A and w^1 at B. All the particles which enter the cylinder at A remain entirely within it, and pass out at B. Let p and v represent the pressure and velocity at A, p^1 and v^1 those at B. Let us now apply the principle of work to the system A B, equating the work done by the outer forces to the increase of energy. The exterior forces are the pressures and gravity. The internal forces do no work, because we suppose a perfect fluid. Of the pressures, only those on the end sections at A and B do any work, because the others are normal to the direction of the motion. Let A B move to A^1 B^1 during a time $d\,t$; then $A\,A^1 = v\,d\,t$; $B\,B^1 = v^1\,d\,t$. The volume of the cylinder $A\,A^1 = w\,v\,d\,t = B\,B^1 = w^1\,v^1\,d\,t$. as the quantities passing A and B in the same time must be equal. Call $w\,v = w^1\,v^1 = \mathrm{Q}$. Then

$$\text{work done by } p = + p\, w\,.\, v\, d\, t = + p\, \mathrm{Q}\, d\, t$$

$$\text{work done by } p^1 = - p^1\, w^1\,.\, v^1\, d\, t\,. = - p^1\, \mathrm{Q}\, d\, t$$

The work done by gravity is the same as that done by transferring the volume A A^1 to B B^1; hence if the height of A and of B, above any fixed plane, be represented by z and z_1, we have

$$\text{work done by gravity} = + \gamma \cdot w \cdot v\, dt \cdot (z - z^1) = + \gamma\, Q\, dt\, (z - z^1)$$

We have now to find the change of energy of the system. A B has moved so as to occupy the position A^1 B^1; but in A^1 B there is at each moment, at any point, a constant velocity, because the motion is permanent. Hence the only difference is that in A B we have A A^1, while in A^1 B^1 we have B B^1 instead; hence the change of energy is that of B B^1 minus that of A A^1; or

$$\text{energy of B } B^1 = \tfrac{1}{2} \cdot \gamma \frac{Q\, dt}{g} \cdot v^{1\,2}$$

$$\text{energy of A } A^1 = \tfrac{1}{2} \frac{\gamma\, Q\, dt}{g} \cdot v^2$$

$$\text{Hence change of energy} = \frac{Q\,\gamma}{2\,g}\, dt\, (v^{1\,2} - v^2)$$

Then we have finally

$$\frac{\gamma}{2\,g} \cdot Q\, dt\, (v^{1\,2} - v^2) = \gamma\, Q\, dt\, (z - z^1) + (p - p^1)\, Q\, dt$$

$$\text{or} \qquad z + \frac{p}{\gamma} + \frac{v^2}{2\,g} = z^1 + \frac{p^1}{\gamma} + \frac{v^{1\,2}}{2\,g},$$

which is Bernouilli's theorem.

This theorem shows that if we follow the same molecule of a mass of liquid in motion, the quantity $z + \frac{p}{\gamma} + \frac{v^2}{z\,g}$ remains forever the same, the motion being permanent, the liquid perfect, and there being no losses of energy such as would be due to shocks or impact. Now $\frac{p}{\gamma}$ represents the height of a column of liquid which would produce a pressure p. If at any point in a liquid, where the pressure is p, we were to insert a vertical tube, open at the bottom, and with a vacuum at the top, the liquid would rise to the height $\frac{p}{\gamma}$; this we call the height due to the pressure. Also, $\frac{v^2}{2\,g}$ is the height due to the velocity. Hence, following the same molecule, the actual height, plus the heights due to the pressure and velocity, make a constant sum. If the liquid is at rest, $z + \frac{p}{\gamma} =$ constant, as in hydrostatics. We call $\frac{p}{\gamma} + \frac{v^2}{2\,g}$ the *head* at the given point. It varies along the path of a given molecule.

If the fluid is not perfect, or if losses of energy occur, we can take account of them as follows: Let z, p, and v refer to a position A of a molecule, and z^1, p^1, v^1, to some point beyond A in the path of the same molecule; let there be losses of energy, or internal work, represented by W, between the two points. Then

$$z + \frac{p}{\gamma} + \frac{v^2}{zg} = z^1 + \frac{p_1}{\gamma} + \frac{v^{12}}{zg} + \mathrm{W}.$$

All that is necessary, therefore, is to find the value of W.

Application of Bernouilli's theorem. — The most frequent problem in hydraulics is to find the velocity or the pressure of water under certain conditions. The point in the liquid which is considered, *i.e.*, the value of z, is considered as known; then the theorem $z + \frac{p}{\gamma} + \frac{v^2}{zg} =$ *constant*, gives us a relation from which, having determined the constant, we may find either v or p, if the other be known. In deciding when we may assume p as known, the following theorems are of value: —

1.° If at any section of a fluid vein all the particles move in parallel straight lines, and with uniform velocities, and if no external circumstances determines a uniform pressure around the circumference of the vein, then the pressure will vary in the vein acccording to the laws of hydrostatics. For the forces of inertia are zero.

2.° If, however, the vein is discharged into the atmosphere, so that the pressure is the same all around the vein, then the pressure is constant throughout the vein.

3.° If the particles of a liquid have any motions, in any directions, the velocities being *very* small, the pressure will vary in the liquid sensibly according to the law of hydrostatics; for the liquid is almost in a condition of rest.

4.° If a mass of liquid be moving through another liquid at rest, and if at any section the particles of the moving liquid be moving in parallel straight lines, normal to the section, and with uniform velocities, then the pressure at that section throughout the entire mass will vary according to the laws of hydrostatics.

5.° To apply Bernouilli's theorem in cases of relative motion we have only to introduce a term expressing the work done by the forces of inertia.

These remarks enable us to find v in many cases, p being known.

In a liquid, p cannot be negative. If the results of theory give negative pressures, it shows that we have made some false hypothesis, or that the motion is not permanent, or cannot take place under the conditions

assumed. Theoretically, so long as it is not negative, the pressure can be as small as you please; but water is charged with air to a considerable extent, and when the pressure falls below the atmospheric pressure, at which the air has been absorbed, it tends to escape. In many cases, therefore, we have in practice to see that p does not fall much below the atmospheric pressure.

Flow of liquids Through Orifices.

Flow through a very small orifice in a thin plate. — Let E F (fig. 19.) be an orifice of very small dimensions in the side or bottom of a vessel. Let a condition of permanent motion be established, the water-level being maintained constant, at a height h above the orifice. Then it is known that the velocity with which the water will be discharged is $\sqrt{2gh}$. This law was announced by Torricelli, in 1643, having been discovered by experiment. We may prove the theorem as follows: Let the thickness of the wall of the vessel be so small that the vein does not touch it except at its inner edge, E F; for this to be true the thickness must be less than half the smallest dimension of the orifice, or else the orifice must be bevelled off to a sharp edge on the inside. Under these circumstances, the liquid particles will converge till they reach $a\ b$, where they move in parallel straight lines. $a\ b$ is the smallest section of the vein, and is called the contracted vein. The orifice being small, the velocity in the contracted vein may be taken as constant. Now, apply the theorem of Bernouilli to a particle passing $a\ b$. This particle came from some point m^1 inside the vessel, and we have

$$z + \frac{p}{\gamma} + \frac{v^2}{2g} = z^1 + \frac{p^1}{\gamma} + \frac{v^{1\,2}}{2g}$$

But $v^1 = o$, and $p^1 = p'' + \gamma\, h'$, because the velocity at m^1 is very small, or zero, hence we find $\frac{v^2}{2g} = h + \frac{p'' - p}{\gamma}$. or

$$v = \sqrt{2g\left(h + \frac{p'' - p}{\gamma}\right)}$$

p'' being the pressure on the surface of the liquid in the vessel, and p that on the contracted vein. If $p'' = p$: $v = \sqrt{2gh}$.

If we wish to be still more accurate, we may prove that

$$v = \sqrt{\frac{2g\left(h + \frac{p'' - p}{\gamma}\right)}{1 - \frac{a^2}{A^2}}}$$

in which a is the area of the orifice and A that of the vessel at the water level.

Let O be the area of $a\ b$: then $\frac{O}{a}$ is called the co-efficient of contraction, μ. The real velocity will be $n\ v$, n being the co-efficient of velocity. The discharge through the orifice, if $p'' = p_1$ will be

$$Q = \mu\ a\ .\ n\ \sqrt{2\ g\ h} = m\ a\ \sqrt{2\ g\ h}\ .$$

m being the co-efficient of discharge. n must be less than unity, and is found by experiment to be from 0.92 to 1.00, and it is generally taken as about 0.97. That it is less than unity is due to losses of head due to friction and other causes. The value of μ is found by theory only in one case; in others it is found by experiment. If fig. 20 represents the jet, then we have the following proportions: —

From Bossut's experiments $a : b : c :: 100 : 81 : 50$

From Michelotti's " $a : b : c :: 100 : 79 : 39.$

so that μ is about 0.64.

Poncelet and Lesbros found the following laws in regard to the co-efficient m: —

1.° For rectangular orifices, m depends upon the smallest dimension, no matter whether that dimension is horizontal or vertical, and is independent of the other dimension, provided it is not over 20 times the first.

2.° The form of the sides, or bottom, of the containing vessel does not affect m, so long as the orifice is removed from them by over 2.7 times the breadth of the orifice.

3.° With a rectangular orifice m depends upon the head h, and becomes larger the smaller the area of the orifice and the smaller h, with some unimportant exceptions.

HYDRAULIC PRESSURE. — The pressure in a liquid in motion is called *hydraulic* pressure, in distinction from *hydrostatic* pressure, in a liquid at rest.

Consider the case shown in fig. 21. It would seem that by increasing the length of the tube indefinitely, if the velocity through the area $g\ h = A^1$ was $\sqrt{2\ g\ h}$, we could increase the discharge indefinitely. This is not so, because in the tube the pressure is less as we ascend, for $z + \frac{v^2}{2\ g}$ is greater than at $g\ h$, hence the pressure is less. The maximum discharge will occur when pressure at $e\ f = o$; then maximum

$$Q = A\ \sqrt{2g\left(h_1 + \frac{p_o}{\gamma}\right)}\ ,$$

and if the discharge through A^1 is to be given by

$$Q = A^1 \sqrt{2 g h}$$

we must have

$$A \sqrt{h_1 + \frac{p_o}{\gamma}} = A^1 \sqrt{h} .$$

1.° Hence if h is a given length

$$A^1 \leqq A \sqrt{\frac{h_1 + \frac{p_o}{\gamma}}{h}} .$$

(a) If A^1 is smaller than this, Q is smaller, and

$$Q = A^1 \sqrt{2 g h}.$$

(b) If A^1 is larger, Q remains the same, or

$$Q = A \sqrt{2 g \left(h_1 + \frac{p_o}{\gamma} \right)}.$$

2.° If $A = A^1$, but h varies, then

$$h \leqq h_1 + \frac{p_o}{\gamma}.$$

(a.) If h is smaller than this, Q is smaller, and

$$Q = . A^1 \sqrt{2 g h} .$$

(b.) If h is larger, Q remains the same, and

$$Q = A \sqrt{2 g \left(h_1 + \frac{p_o}{\gamma} \right)}$$

In both the cases (*b*), the water discharges through A into a vacuum, and the velocity through A^1 is less than $\sqrt{2 g h}$. A new water level will form in the tube, at a height x, sufficient to carry off the quantity Q through A^1; thus

$$\text{max. } Q = A \sqrt{2 g \left(h_1 + \frac{p_o}{\gamma} \right)} = A^1 \sqrt{\frac{2 g \left(x - \frac{p_o}{\gamma} \right)}{1 - \left(\frac{A}{A_1} \right)^2}}$$

If $A = A'$, then $x = \frac{p_o}{\gamma}$.

We have supposed in this case, as shown by the figure, that the orifice at the top were rounded so that the co-efficient of contraction is 1. The reason of this will soon be seen.

HYDRAULIC EXPERIMENTS. — Experiments to determine the values of the co-efficients described are made as follows: The water is discharged from an orifice and allowed to flow continually, the quantity discharged being measured in some measuring vessel. By finding Q, and measuring the area of the contracted vein, the value of v is found; v may also be found, with a small orifice, by observing the vein, and measuring the two co-ordinates of some point upon it.

INVERSION OF THE VEIN. —When the orifice is not very small, the particles issuing do not all have the same velocity, those coming from the top having less velocity than those from the bottom. Hence they tend to pursue different trajectories, and the paths of the different particles intersect, as shown in figure 21. This gives rise to a distortion in the shape of the vein, known as the inversion of the vein. If the orifice is a square, the shape of the vein at a certain distance from the orifice becomes a square, whose sides make angles of 45° with those of the orifice. Poncelet and Lesbros found other and very curious shapes.

DISCHARGE FROM LARGE ORIFICES. —When the orifice becomes so large that the velocities of different particles would be sensibly different, we must take account of this variation in our expression for the discharge. We consider several cases.

1.° *Rectangular orifice* (Fig. 22). — The breadth being b, the discharge from any small horizontal strip $b\,dy$ at a distance $H + y$ from the surface of the water will be

$$dQ = m \cdot b\,dy\ \sqrt{2g\,(H + y)}$$

$$\therefore\ Q = \tfrac{2}{3}\, m\, b\ \sqrt{2g}\left[(H + a)^{3/2} - H^{3/2}\right]. \tag{1.}$$

The discharge may also be obtained by assuming that the average velocity through the orifice will be the velocity at the centre of gravity. This gives

$$Q^1 = m\, b\, a\ \sqrt{2g}\ \sqrt{\left(H + \frac{a}{2}\right)}. \tag{2.}$$

The ratio of these two is

$$\frac{Q}{Q^1} = \tfrac{2}{3}\,\frac{(H + a)^{3/2} - H^{3/2}}{a\ \left(H + \frac{a}{2}\right)^{\frac{1}{2}}}.$$

Or for different values of $\frac{H}{H+a}$, as follows:—

$\frac{H}{H+a}$ =	0.0	0.2	0.4	0.6	0.8	1.00
$\frac{Q}{Q^1}$ =	0.943	0.974	0.992	0.997	0.999	1.000

This shows that the two formulæ give results almost identical, and never over 6 per cent different. Hence one is just as good as the other for practical use, for the suppositions made in deducing (1) are not fulfilled actually, and it can claim no more accuracy in reality than (2). We may therefore put $Q = m\,a\,\sqrt{2gz}$, where z is the depth of the centre of gravity of the orifice

2.° *Triangular orifice* (Fig. 23). $b = y.\frac{l}{a}$. $\therefore$

$$dQ = m.\,y\,\frac{l}{a}\,.\,dy\sqrt{2g\,(H+y)}$$

$$Q = \frac{l}{a}\,m\sqrt{2g}\left[\tfrac{2}{3}\,a\,(H+a)^{3/2} - \frac{4}{15}(H+a)^{5/2} + \frac{4}{15}\,H^{5/2}\right]$$

3.° *Triangular orifice* (Fig. 24). $b = (l-y)\frac{l}{a}$.

$$dQ = m.\,(a-y)\frac{l}{a}\,.\,dx\sqrt{2\,g\,(H+y)}$$

$$Q = \frac{l}{a}\,.m\sqrt{2\,g}\left[\frac{4}{15}\,(H+a)^{5/2} - \frac{4}{5}\,H^{5/2} - \tfrac{2}{3}\,a\,H^{3/2}\right]$$

These equations will be referred to again. For the practical calculation of discharge through triangular orifices the equation $Q = m\,a\,\sqrt{2gz}$ may be used, and in fact for orifices of any shape. Hachette's experiments, 1805, showed that the discharge through orifices was practically independent of their shape, so long as there were no re-entrant angles.

4.° *Circular orifices.* — The exact expression for the discharge may be found as in the previous cases, although in this case the integration results in an infinite series. It may be shown, however, that the greatest difference between the results of the formula obtained, and those given by the formula taking the average velocity at the centre of gravity, is not over 4 per cent. Hence we may always use the ordinary formula.

$$Q = m\,A\,\sqrt{2\,g\,z}.$$

Co-efficients. —We have see that $m = \mu\,n$, hence the co-efficient of discharge equals the product of the co-efficients of velocity and of contraction. For small orifices we saw that approximately $\mu = .64$ and $n = 0.97$, hence $m = \cdot 91$ very nearly.

For *rectangular* orifices, the results of Poncelet and Lesbros' experiments may be used, as given in the tables appended: —

VALUES OF m IN THE FORMULA

$$Q = m\,A\sqrt{2g\left(H + \frac{h}{2}\right)}$$

H being depth of top of orifice, and h the height of orifice.

From Poncelet and Lesbros.

Rectangular vertical orifice in a thin plate, with complete contraction; width of orifice, 8 inches.

Head above top of orifice H*	Co-efficient m for height of orifice. (Inches.)					
	8	4	2	1.2	0.8	0.4
Inches.						
0.2	–	–	–	–	–	0.705
0.4	–	–	0.606	0.629	0.659	0.700
0.6	–	0.592	0.611	0.631	0.659	0.696
0.8	0.571	0.595	0.614	0.633	0.658	0.693
1.0	0.574	0.597	0.617	0.635	0.658	0.690
1.5	0.580	0.602	0.622	0.639	0.658	0.684
2.0	0.584	0.604	0.624	0.639	0.657	0.678
4.0	0.591	0.610	0.630	0.636	0.654	0.666
8.0	0.597	0.614	0.630	0.632	0.648	0.655
12.0	0.600	0.616	0.629	0.632	0.644	0.650
16.0	0.601	0.617	0.628	0.631	0.642	0.647
24.0	0.604	0.617	0.627	0.630	0.638	0.642
40.0	0.605	0.615	0.626	0.628	0.633	0.632
60.0	0.602	0.611	0.620	0.620	0.619	0.615
80.0	0.601	0.607	0.613	0.612	0.612	0.611
120.0	0.601	0.603	0.606	0.608	0.610	0.609

* Head measured at a point in reservoir where water is absolutely quiet.

Values of m in the Formula

$$Q = m\sqrt{2gz}.$$

From Ellis' Experiments.

Kind of Orifice.	Head on Centre.	Co-efficient.
	feet.	
Vertical Orifice, 2′ horizontal by 1.99975′ vertical.	2.069	0.610
	3.049	0.597
	3.541	0.606
Vertical Orifice, 2′ horizontal by 1′ vertical.	1.813	0.597
	3.035	0.599
	6.866	0.598
	8.476	0.599
	11.314	0.605
Vertical Orifice, 2′ horizontal by ½′ vertical.	1.423	0.611
	2.905	0.611
	6.356	0.608
	11.563	0.604
	16.965	0.600
Vertical Orifice, one foot square.	1.487	0.585
	3.699	0.598
	6.769	0.599
	9.863	0.599
	12.005	0.600
	15.132	0.601
	17.565	0.597

For *circular* orifices, the following tables will serve as a guide: —

Diameter.		Head on Centre.		Co-efficient.	Authority.
0.4	inches.	25.6	inches.	0.628	Weisbach.
0.4	"	9.84	"	0.637	"
0.8	"	25.6	"	0.621	"
0.8	"	9.84	"	0.629	"
1.2	"	25.6	"	0.614	"
1.2	"	9.84	"	0.622	"
1.6	"	25.6	"	0.607	"
1.6	"	9.84	"	0.614	"
1.06	"	12.5	feet.	0.616	Bossut.
2.13	"	12.5	"	0.618	"
6	"	2.15	"	0.599	Ellis.
6	"	6.35	"	0.604	"
6	"	10.51	"	0.601	"
6	"	14.47	"	0.601	"
6	"	17.26	"	0.596	"
12	"	1.15	"	0.574	"
12	"	4.82	"	0.590	"
12	"	10.89	"	0.594	"
12	"	14.13	"	0.595	"
12	"	17.73	"	0.600	"
24	"	1.77	"	0.589	"
24	"	4.48	"	0.603	"
24	"	5.84	"	0.609	"
24	"	8.35	"	0.612	"
24	"	9.64	"	0.615	"

For other shapes of orifice these examples must serve as a guide in the absence of reliable experiments.

THEORETICAL DETERMINATION OF μ. — Navier has given an ingenious theoretical determination of μ which gives a result remarkably near that found by experiment, although his suppositions are arbitrary. He supposes that all the particles pass the plane of the orifice with the same velocity with which they pass the contracted vein, *i.e.*, with the velocity $\sqrt{2\,g\,(z+\frac{p-p'}{\gamma}}$. The angle which the direction of their motion, however, makes with the plane of the orifice will vary, and he assumes it to vary from 0° at the centre to 90° at the edges. Let $d\,a$ be the element of area which is passed by a liquid particle with a velocity v, at an angle β with the plane of the orifice; then the quantity passing in a unit of time will be

$$d\,Q = d\,a.\ v.\ \sin.\ \beta. \qquad \text{and}$$

$$Q = \int d\,a.\ v\ \sin.\ \beta = v \int \sin.\ \beta.\ d\,a\,.$$

Now, there are an infinite number of elements $d\,a$, and an infinite number of angles β. If $d\,a$ represents the entire area over which the angle β is the same, then each β corresponds to a certain $d\,a$, though there will still be an infinite number of elements $d\,a$, and the same number of angles β. Now, if 90° be divided into the number of parts that there are angles β, to each β will correspond a $d\,a$, and we may write

$$d\,a\ :\ a\ ::\ d\,\beta\ :\ \frac{\pi}{2}\ \therefore\ d\,a = \frac{2\,a}{\pi}\,d\,\beta$$

a being the entire area of the orifice. Inserting this in the equation, and integrating, we obtain

$$Q = \frac{2\,a\,v}{\pi} \int_0^{\frac{\pi}{2}} \sin.\ \beta.\ d\,\beta = 0.637\ a\,v\,.$$

Hence $\mu = 0.637$.

The co-efficients given apply only to orifices in a thin plate, with complete contraction, and no mouthpiece of any kind. We have seen that the average co-efficient is thus about .61, that we may alter this value by altering the character of the orifice so as to change the value of the contraction, increasing it or decreasing it. We will now show how we may decrease the co-efficient of *contraction* to 0.5, and explain at the same time this remarkable case, which allows of a theoretical determination of that co-efficient.

BORDA'S MOUTHPIECE (Fig. 25). – Let E F be a small orifice in the side K L of the vessel, in which the water is kept at a constant level C D; and let the re-entering pipe or mouthpiece A B F E be fitted to the orifice as

shown. The length A C must be so short that the vein shall not touch B F after leaving B, or A E should not be greater than A B. If the orifice were on the bottom of the vessel the distance A E could be greater. Now, in this case, the particles along the entire wall D E and F L, even at E and F, have small velocities, which would not be the case were the mouthpiece removed. Hence, the pressure on the wall D L may be considered to vary sensibly according to the law of hydrostatics, and those pressures will just balance those on the opposite wall C M, except the pressure acting on O P, which is the orifice A B projected across horizontally. Now assume a horizontal axis of reference, and apply to the mass of liquid between C D and $c\ d$ (the contracted vein) the equation expressing the variation of momentum during the time $d\ t$. In this time the mass C D $c\ d$ passes to C′ D′ $c'\ d'$. The portion C′ D′ $c\ d$ is occupied during the time by a varying mass of liquid, at every point of which the condition of things remains exactly the same constantly, because we are considering permanent motion. In order to find the change of momentum of the mass, therefore, we must only recollect that at the beginning of the time we have the volume C D C′ D′, while at the end of the time we have replaced this by $c\ d\ c'\ d'$. Now the volume $c\ d\ c'\ d'$ is O $v.\ d\ t$ (O being the area of the contracted vein); its mass is $\frac{\gamma\ \mathrm{O}\ v\ d\ t}{g}$; and this is also the mass of C D C′ D′. The momentum of C D C′ D′, however, with reference to a horizontal axis, is zero, as its motion is vertical; that of $c\ d\ c'\ d'$ is $\frac{\gamma\ \mathrm{O}\ v^2\ d\ t}{g}$, and this is the change of momentum of the entire mass C D C′ D′ in the time $d\ t$. The forces acting on the mass are gravity, the pressures on the sides of the vessel, and the atmospheric pressures on C D and A B. As the vertical forces have no component along the axis assumed, we have only to consider the horizontal forces. The pressures on the side of the vessel are balanced, except that on O P, which is $(p_o + \gamma\ h)\ a$, and it acts toward the right, as we must consider the forces exerted *on* the mass of water. The atmospheric pressure on A B is $p_o\ a$, and it acts toward the left, or opposed to the direction of motion. Hence we have the equation

$$\frac{\gamma\ \mathrm{O}\ v^2\ d\ t}{g} = \Big[(p_o + \gamma\ h)\ a - p_o\ a\ \Big] d\ t$$

$$= \gamma\ h\ a\ d\ t;\ \text{hence}$$

$$\frac{\mathrm{O}\ v^2}{g} = h\ a\ ;\ \text{but}\ \frac{v^2}{2\ g} = \ \therefore\ \frac{v^2}{g} = 2\ h\ \therefore$$

$$2\ \mathrm{O} = a$$

$$\frac{\mathrm{O}}{a} = \mu = \tfrac{1}{2}\ .$$

This demonstration may be extended to the case of inclined surfaces (fig. 26), as follows: We will only outline the steps in this case, which is exactly similar to the previous one. We take as axis a line perpendicular to E F, and through its centre of gravity, which will also be approximately the centre of gravity of the contracted vein. Consider a mass of liquid between $c\ d$ and a circular cylinder M N P Q described about the axis of reference, and large enough so that on all its surfaces the liquid has small velocities, except at the orifice. Then, as before, the change of momentum reduces to $\frac{\gamma\ O\ v^2\ d\ t}{g}$, if we can neglect the velocities in P Q. Those in Q M, P N are at right angles to the axis. Let S be the area P Q, G the centre of gravity of F E, and h its vertical depth below the surface A B; also let z be the depth of O. Then the component of gravity is $+$ S. O G. $\gamma\ d\ t.$ cos. $a = (h - z)$ S. $\gamma\ d\ t.$ The pressures on Q M and P N have no component. That on P Q has a component $+ (p_o + \gamma\ z)$ S; that on E M and F N has $- (p_o + \gamma\ h)\,(S - a)$; that of the atmosphere on A B has $- p_o\ a$; hence we have

$$\frac{\gamma\ O\ v^2\ d\ t}{g} = (h - z)\ S.\ \gamma\ d\ t + (p_o + \gamma\ z)\ S\ d\ t - (p_o + \gamma\ h)\,(S - a)\ d\ t - p_o\ a\ d\ t$$

$$= g\ h\ a\ d\ t$$

$$\frac{O\ v^2}{g} = h\ a = 2\ h\ O \quad \therefore \quad \frac{O}{a} = \tfrac{1}{2}, \text{ as before.}$$

In this case the co-efficient of contraction is about $\frac{1}{2}$. In the case of a simple orifice, the particles move along the wall at E and F with appreciable velocities, so that the pressures at those points do not balance those on the opposite wall. The force acting to increase the momentum is, therefore, increased, or the right hand side of the equation becomes larger; hence, instead of having $\frac{\gamma\ O\ v^2\ d\ t}{g} = \gamma\ h\ a\ d\ t$, we have

$$\frac{\gamma\ O\ v^2\ d\ t}{g} > \gamma\ h\ a\ d\ t, \quad \text{or} \quad \frac{O}{a} > \tfrac{1}{2}, \text{ as we know to be the case.}$$

Incomplete contraction occurs when the vein touches the sides of the orifice after leaving its inner edge (where the walls of the vessel are thick, for instance), when the orifice is not in a plane surface, or when contraction is partially suppressed by a plate extending inward from some part of the circumference of the orifice. The effect of incomplete contraction is to increase the co-efficient, except in cases analogous to Borda's orifice, and which really do not come under this head.

If the orifice be made in a thick plate, and of the shape of the contracted vein, or if a mouthpiece be fitted to it, having that shape, the co-efficient of discharge from the outer orifice will be μ, as there will be no

co-efficient of contraction, or $m = .97$. We see, therefore, that by varying the arrangement of the orifice, we may vary m between the values of 0.50 and 0.97.

Lesbros has studied the discharge through ordinary sluices in which the walls were about two inches thick, and the contraction partial. He found m to vary with different arrangements between 0.59 and 0.71, the lower figure occurring when the contraction was nearly perfect.

For cases where contraction is suppressed on one side, as shown in fig. 27, Lesbros found that the co-efficient of discharge is increased, not in proportion to the number of sides on which contraction is suppressed, but in the ratio of the fraction of the total perimeter of the orifice on which contraction is suppressed; and, other things equal, the increase is greatest when the base is among the sides where contraction is suppressed. Bidone gave the formula $m_1 = m\left(1 + A\frac{n}{p}\right)$ in which m_1 is the co-efficient for partial and m for complete contraction under the same conditions, p the total perimeter of the orifice, and n the part of the perimeter on which the contraction is suppressed. A is a constant which Bidone gives as

$A = 0.1523$ for rectangular orifices.

$A = 0.1280$ for circular orifices.

Weisbach finds from his experiments

$A = 0.1343$ for rectangular orifices.

In using these formulæ, however, it is essential that $\frac{n}{p}$ shall not approach too nearly to unity, or else we shall approach the condition of Borda's mouthpiece, or another case to be subsequently treated. The formula of Bidone does not agree very well with Lesbros' experiments, the results of which are given in the tables on the following page.

Effect of inclined guide. — If the water is guided to the orifice by an inclined plate at its upper side, as in fig. 28, as is often the case where water is admitted to water-wheels by an inclined gate, the contraction will be partial, and the co-efficient increased. Experiments on this point are very few, but it is clear that the co-efficient will be greater the greater the angle α. The following rule agrees with some experiments by Poncelet: find Q as though the orifice were vertical, and multiply by $1 + 0.47 \sin. \alpha$. This is only a rough rule, and should only be used when $\alpha < 45.''$

Co-efficient of Discharge for Partial Contraction.

(Lesbros.)

Rectangular orifice in thin plate, 8″ wide, and of various heights, discharging freely into air.

	Head on top of orifice in inches.	Co-efficients for various heights of orifice. (Inches.)					
		8	4	2	1.2	0.8	0.4
Contraction suppressed on lower side.	0.8	0.598	0.623	0.663	0.691	0.703	0.754
	1.0	0.600	0.625	0.664	0.688	0.702	0.750
	1.5	0.604	0.631	0.665	0.686	0.700	0.743
	2.0	0.607	0.634	0.666	0.685	0.699	0.735
	4.0	0.614	0.642	0.668	0.683	0.697	0.721
	8.0	0.620	0.647	0.670	0.680	0.695	0.711
	12.0	0.621	0.647	0.669	0.680	0.694	0.708
	16.0	0.622	0.647	0.668	0.680	0.694	0.705
	24.0	0.623	0.647	0.668	0.678	0.693	0.703
	40.0	0.623	0.646	0.666	0.675	0.692	0.701
	60.0	0.623	0.643	0.664	0.674	0.687	0.667
	80.0	0.618	0.640	0.663	0.674	0.682	0.692
	120.0	0.613	0.638	0.661	0.674	0.679	0.688
Contraction suppressed on vertical sides.	0.8	..	..	0.655	..	..	0.715
	1.0	..	..	0.654	..	..	0.710
	1.5	..	..	0.651	..	..	0.701
	2.0	0.648	..	0.649	..	..	0·695
	4.0	0.645	..	0.645	..	..	0.683
	8.0	0.641	..	0.642	..	..	0.675
	12.0	0.639	..	0.642	..	..	0.671
	16.0	0.639	..	0.641	..	..	0.668
	24.0	0.638	..	0.637	..	..	0.665
	40.0	0.638	..	0.631	..	..	0.658
	60.0	0.637	..	0.627	..	..	0.651
	80.0	0.636	..	0.621	..	..	0.647
	120.0	0.634	..	0.614	..	..	0.644
Contrac'n suppressed on lower and vertical sides.	2.0	..	..	0.700	..	..	..
	4.0	..	..	0.696	..	..	..
	8.0	0.708	..	0.693	..	..	..
	12.0	0.687	..	0.691	..	..	..
	16.0	0.682	..	0.690	..	..	..
	24.0	0.679	..	0.688	..	..	..
	40.0	0.676	..	0.685	..	..	..
	60.0	0.672	..	0.681	..	..	..
	80.0	0.668	..	0.680	..	..	..
	120.0	0.665	..	0.678	..	..	..

Loss of energy of vein. — The theoretical velocity with which the vein should issue from an orifice is $v_1 = \sqrt{2\,g\,h}$. The real velocity is $n\,v$. The theoretical energy of a mass of water m issuing from the orifice, and its actual energy, are

$$\text{Theoretical energy} = \tfrac{1}{2}\,m\,v^2.$$
$$\text{Actual energy} = \tfrac{1}{2}\,m.\ n^2\,v^2.$$

Hence, loss of energy $= \tfrac{1}{2}\,m\,v^2\,(1 - n^2)$. Taking $n = 0.97$ we have $1 - n^2 = 0.06$. Hence, the loss of energy is 6 per cent of the total energy of the water. This is independent of the co-efficient of contraction.

SUBMERGED ORIFICES. — An orifice is submerged if, instead of discharging freely into the air, so that the air surrounds the contracted vein, it discharges into a channel or basin in which the water stands at a level above the bottom of the orifice. If it stands above the top of the orifice the latter is *completely submerged;* if it stands at a level between the top and bottom, the orifice is *partially submerged.* We distinguish three cases, shown in figs. 29, 30, and 31.

Case I (Fig. 29). — Applying the theorem of Bernouilli between the upper reservoir and the contracted vein, we find at once

$$v = \sqrt{2\,g\,\frac{\mathrm{H} + \frac{p - p_o}{\gamma}}{1 - \left(\frac{a'}{a}\right)^2}}$$

or, generally, $v = \sqrt{2\,g\,\mathrm{H}}.$

$$\mathrm{Q} = m\,a\,\sqrt{2\,g\,\mathrm{H}}.$$

Case II (Fig. 30). — The same formulæ apply to this case.

The value of m will of course depend upon the form and arrangement of the orifice. If a mouthpiece having the shape of the contracted vein be used, then $m = 0.97$ approximately.

Case III (Fig. 31). — We may consider the discharge here to be intermediate between that into free air and that in Case II. In fact, we may consider the orifice in two parts; first, $a\ b$, the discharge through which we consider as the same as into free air; and, second, $b\ c$, in which we consider the discharge the same as in Case II. We thus arrive at the formula

$$\mathrm{Q} = m_1\,b\,h_1\,\sqrt{2\,g\,\mathrm{H}_1} + m_2\,b\,(h - h_1)\,\sqrt{2\,g\left(\mathrm{H}_2 + \frac{h - h_1}{2}\right)}$$

for the most common cases.

Unfortunately the values of m_1 and m_2 are not known.

Regarding Cases I. and II., Bornemann deduced the following formula from his experiments (see fig. 32): —

$$Q = \left(0.63775 + 0.30\,\frac{e}{h_3}\right) b\, e\, \sqrt{2\,g\,(h_1 - h_2)}$$

in which $h_3 = h_2 - \frac{e}{2}$.

In these experiments, however, the contraction was probably not complete, and the co-efficient therefore too large.

Bornemann has given a later formula, which he considers more exact, as follows: —

$$Q = \left[0.43479 + 0.25666\sqrt{\frac{e}{h_1 + \frac{e}{2}}} + 0.03121\,\frac{1}{h_2 + \frac{e}{2}}\sqrt{\frac{1}{b}}\right] b\, e \sqrt{2\,g(h_1 - h_2)}$$

This value of m, however, is too complicated for ordinary use.

In Case II. the usual procedure is to calculate just as though the discharge was into open air, and then use a co-efficient. Lesbros has given a table, from his experiments, which may be used.

Ellis found for circular and square submerged orifices, under varying heads, a co-efficient in the formula $a\sqrt{2\,g\,H}$ almost constant, and $= 0.602$.

Losses of head. — We have hitherto considered cases in which there were no losses of head, and in which Bernouilli's theorem had the form $z + \frac{p}{\gamma} + \frac{v_2}{2\,g} = \text{constant}$. We have now to consider what losses of head may occur. These losses are of three kinds. (1.) Losses from the internal friction of the molecules, due to the fact that adjacent particles do not move with exactly the same velocities, and that we have to deal with a fluid which is not perfect. This loss always occurs, and as we do not know the laws of distribution of velocity among the particles, nor the co-efficient of velocity, we cannot take account of it in our calculations. Some mathematicians, however, by means of certain suppositions, have discussed the subject in a general way, but the results are not of practical utility. It is this loss, together with the second, which causes the real velocity to be less than $\sqrt{2\,g\,h}$ in the case of discharge through an orifice.

(2.) Losses due to the friction of the liquid on the surfaces containing it, as against the sides and bottom of a canal, or the circumference of a pipe. This loss is a very important one in cases where the distances considered are large. For small distances, such as we are now consider-

ing, it may be neglected. We shall consider its laws later, in connection with the flow in channels and pipes.

(3.) Losses due to sudden reductions in the velocity of the liquid, such as occurs when a pipe is suddenly enlarged at a certain point, and the velocity of the liquid flowing in it correspondingly diminished. This loss is analogous to that occurring in the case of a shock, or the impact of two elastic bodies. This case must now be considered.

Effect of an abrupt change of section in a closed tube (Fig. 33). — Suppose a liquid to pass from the tube A B C D through the orifice G F into the tube L M K I. Let E H be the contracted vein, and I K a plane where the particles are again supposed moving in parallel straight lines normal to the section. The space F H K_1 M and G E I_1 L is filled with a mass of liquid not participating in the general motion, but sensibly stagnant, although probably more or less agitated and possessed of an eddying or rotary motion. In this case it is clear that some work is done, and some loss of energy caused, and before we can apply the theorem of Bernouilli to two points between which such a loss occurs, we must first find the value of the loss. In order to solve this problem, we must make the following reasonable suppositions, and afterwards we may test our results experimentally.

(1.) According to what has been said regarding slow velocities, we assume the pressure in the mass of liquid F H K_1 M — G E I_1 L to vary according to the law of hydrostatics, so that if we should put a piezometer in at any point of this mass, or in the section E H_1 the water would rise to the same height in both.

(2.) The pressure in the section I K varies according to the laws of hydrostatics. Now, let two tubes, Q and P, be put down, as shown in the figure, and let h be the height of the water in P above that in Q; θ = loss of head; v_o = velocity in E H; v = that in I K; α = the angle between O V_o and O V, which represent these velocities in magnitude and direction; U = the velocity represented by V V_o; S = area of I K. Now apply the theorem of Bernouilli to a particle passing E H and I K, and we have plain letters referring to E H and primed letters to I K.

$$z + \frac{p}{\gamma} + \frac{v_o^2}{2g} = z' + \frac{p'}{\gamma} + \frac{v^2}{2g} + \theta$$

$$\text{or} \qquad \left(z' + \frac{p'}{\gamma}\right) - \left(z + \frac{p}{\gamma}\right) = \frac{v_o^2}{2g} - \frac{v^2}{2g} - \theta .$$

But the first term of this equation is h. $\therefore\ h + \theta = \frac{v_o^2}{2g} - \frac{v^2}{2g}$.

In order to find θ, we must find h. For this purpose apply the theorem of change of momentum (which we have seen eliminates the action of in-

ternal forces) to the mass of liquid L G E H F M K I. In a time $d\,t$ this mass changes to L G E^1 H^1 F M K^1 I^1, and the change of momentum is the momentum of I K K^1 I^1 minus that of E H H^1 E^1. The volume I K K^1 I^1 is S $d\,v\,t$, hence the change of momentum is $\frac{S\,v\,d\,t}{g}(v - v_o \cos.\ a)$, our axis being taken parallel to that of the cylinder L K. We have now to find the impulse of the outer forces, which may be found as follows: suppose, for an instant, the liquid at rest, then the outer forces are all in equilibrium, hence their projection on any axis is zero. Now, when motion takes place, the only difference in the state of things, so far as it affects the equilibrium of the outer forces, is that on the plane I K the pressure is that due to a hydrostatic level h higher than that on L M, while before the hydrostatic levels were equal. Hence the value of (force $\times$ time) is $-\gamma\,h.\ S\,d\,t$, because P is supposed higher than Q. [A more rigid proof of this may easily be given.] Hence we have the equation

$$-\frac{\gamma\,S\,v\,d\,t}{g}(v - v_o \cos.\ a) = -\gamma\,S\,h\,d\,t \quad \therefore$$

$$h = \frac{v\,(v_o \cos.\ a) - v_o)}{g}.$$

Substituting this in the equation for θ,

$$\theta = \frac{1}{2g}\left(v_o^2 - v^2 - 2\,v\,v_o \cos.\ a + 2\,v^2\right)$$

$$= \frac{1}{2g}\left(v_o^2 + v^2 - 2\,v\,v_o \cos.\ a\right) = \frac{U^2}{2g}.$$

The velocity U is that which must be compounded with v to produce a resultant v_o, hence *the loss of head is equal to the head due to the geometrical diminution of velocity.*

$$\text{If } a = o\ ;\ \theta = \frac{(v_o - v)^2}{2g}.$$

We have $Q = S\,v = m\,S_o\,v_o \quad \therefore$

$$\theta = \frac{Q^2}{2g}\left(\frac{1}{m_o\,S_o} - \frac{1}{S}\right)^2$$

S being the area of the orifice G F.

Application. — Let fig. 34 represent a short tube attached to a reservoir, and divided into compartments by the diaphragms as shown. Let the areas of the orifices, commencing at the reservoirs, be A, A_2, A_2 . . . . . A_n. Let the areas of the different sections of the tube, commencing at the reservoir, be S, S_1, S_2, ... S_{n-1}. Let h be the difference of level between the water in the reservoir and the centre of gravity of the last

orifice. Required: the quantity Q discharged per second, and the velocity V at the last orifice. The theorem of Bernouilli applied between the reservoirs and A_n gives

$$z + \frac{p_o}{\gamma} + 0 = z' + \frac{p'}{\gamma} + \frac{V^2}{2g} + \theta, \quad \text{or}$$

$$\frac{V^2}{2g} = \left(h + \frac{p_o - p'}{\gamma} \right) - \theta \ldots\ldots (1)$$

If the tube is so short that we may neglect friction on its sides, we shall have

$$\theta = \frac{Q^2}{2g}\left[\left(\frac{1}{m\,A} - \frac{1}{S}\right)^2 + \left(\frac{1}{m\,A_1} - \frac{1}{S_1}\right)^2 + \left(\frac{1}{m\,A_2} - \frac{1}{S_2}\right)^2 + \ldots \right] \ldots (2)$$

$$Q = m\,A_n\,V \ldots\ldots (3)$$

From these three equations we may find the three unknown quantities, Q, V, θ.

Thus eliminating V, we have, calling $p_o = p'$,

$$\frac{Q^2}{2\,g\,m_2\,A_n^2} = h - \frac{Q^2}{2g}\left[\Sigma\left(\frac{1}{m\,A} - \frac{1}{S}\right)^2\right] \quad \text{or}$$

$$Q^2 = \frac{2\,g\,h}{\frac{1}{m^2\,A_n^2} + \Sigma\left(\frac{1}{m\,A} - \frac{1}{S}\right)}$$

$$V^2 = \frac{2\,g\,h}{1 + m^2\,A_n^2\,\Sigma\left(\frac{1}{m\,A} - \frac{1}{S}\right)^2}$$

The above formulæ afford an excellent means of testing the accuracy of the value we have found for θ. Eytelwein took a circular tube 0.0262^m. in diameter and 0.942^m. long, divided by two intermediate diaphragms into three chambers. The areas in the diaphragms and in the ends were 0.00655^m in diameter. Hence we should have

$$\frac{Q^2}{2g} = \frac{h}{\frac{1}{m^2\,A^2} + 3\left(\frac{1}{m\,A} - \frac{1}{S}\right)^2} = \frac{m^2\,A^2\,h}{1 + 3\left(1 - \frac{m\,A}{S}\right)^2}$$

$$Q = A\sqrt{2\,g\,h}\;\frac{m}{\sqrt{1 + 3\left(1 - \frac{m\,A}{S}\right)^2}}$$

Taking $m = 0.62$, and $\frac{A}{S} = \frac{1}{16}$, we have

$$Q = 0.319\ A \sqrt{2\,g\,h}.$$ while experiment gave

$$Q = 0.331\ A \sqrt{2\,g\,h}.$$

This experiment is close enough, considering the uncertainty in m, to show that our assumptions in deducing θ are admissible. Moreover, it was found in the above case that the diaphragms were too close together, so that there was not room between them for the liquid to attain a motion in parallel straight lines filling the entire tube. Hence the co-efficient found was too large; and experiment showed that it decreased as the diaphragms were moved farther apart.

Loss of head between two reservoirs. — Let fig. 35 represent two communicating reservoirs, in which the levels are kept constant. Then, if we neglect the small velocities at the points A and B, we have $\theta = \frac{v^2}{2g}$, and by applying Bernouilli's theorem between those points we shall find $V = \sqrt{2\,g\left(z + \frac{p - p'}{\gamma}\right)}$ as we have already found in another way.

Let fig. 36 represent a reservoir with two partitions, and let it be required to find the discharge Q and the velocity V from the last compartment P. Let a_1, a_2, a_3, etc., $\ldots\ a_n$ represent the area of the orifices in the partitions, supposing that there may be any number: x_1, x_2, x_3, etc., the differences of level between successive compartments, and h the distance of the centre of gravity of the last orifice below the upper level.

Then

$$Q = m\,a_1 \sqrt{2\,g\,x_1}$$

$$Q = m\,a_2 \sqrt{2\,g\,x_2}$$

etc.

$$Q = m\,a_n \sqrt{2\,g\,(h - x_1 - x_2 \ldots .)}$$

Hence

$$x_1 = \frac{Q^2}{2\,g\,m^2\,a_1^2}$$

$$x^2 = \frac{Q^2}{2\,g\,m^2\,a_2^2}$$

. . . . . .

$$h - x_1 - x_2 - \ldots = \frac{Q^2}{2\,g\,m_2\,a_n^{\,2}} \quad \therefore$$

$$h - \frac{Q^2}{2\,g\,m^2\,a_1^2} - \frac{Q^2}{2\,g\,m^2\,a_2^2} - \ldots - = \frac{Q^2}{2\,g\,m^2\,a_n^{\,2}}$$

$$h = \frac{Q^2}{2g} \Sigma \frac{1}{m^2 a^2} \; ; \quad Q = \sqrt{\frac{2gh}{\Sigma \frac{1}{m^2 a^2}}}$$

EFFECT OF VELOCITY OF APPROACH UPON CO-EFFICIENT OF DISCHARGE. — In finding the discharge through any orifice, as the one at the extremity of the tube in fig. 37, we apply Bernouilli's theorem to a point at the orifice, and in the reservoir behind it, where v and p are known. The discharge which results is, therefore, theoretically independent of the velocity in the tube A B. Weisbach has found, however, that the co-efficient m in such a case is affected by that velocity, as it is affected by the proximity of walls, or the supression of contraction. Let $\frac{a}{A} = x =$ ratio of area of orifice to that of vessel just above orifice. Then, according to Weisbach,

$m' = m\left[1 + 0.04564\ (14.821^x - 1)\right] = m \curlywedge$ for circular orifices.

$m' = m\left[1 + 0.076\ (9^x - 1)\right]$ for rectangular orifices.

The use of the formula may be dispensed with, and the following tables used instead: —

I. CIRCULAR ORIFICES.

x	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$\curlywedge$	1.007	1.014	1.023	1.034	1.045	1.059	1.075	1.092	1.112	1.134
x	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
$\curlywedge$	1.161	1.189	1.223	1.260	1.303	1.351	1.408	1.471	1.546	1.613

II. RECTANGULAR ORIFICES.

x	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$\curlywedge$	1.009	1.019	1.030	1.042	1.056	1.071	1.088	1.107	1.128	1.152
x	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
$\curlywedge$	1.178	1.208	1.241	1.278	1.319	1.365	1.416	1.473	1.537	1.608

In cases where the open vessel, or reservoir, is itself so small that there is an appreciable velocity, the formula for discharge through an orifice in it becomes, as we have seen,

$$Q = m\ a \sqrt{\frac{2\ g\left(h + \frac{p - p'}{\gamma}\right)}{1 - \frac{a^2}{A^2}}} \quad \text{or} \quad m' = \frac{m}{\sqrt{1 - \frac{a^2}{A^2}}}.$$

Weisbach has found that the co-efficient m' is really larger than found by this formula, and is given by the equation

$$m' = k\ m = m\left[1 + 0.641\ \frac{a^2}{A^2}\right]$$

$\frac{a}{A}$	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
K	1.002	1.006	1.014	1.026	1.040	1.058	1.079	1.103	1.130	1.160

The ratio $\frac{a}{A}$ should not be much over $\frac{1}{20}$ for the proper use of this formula.

Cylindrical mouthpiece. — If an orifice in a thin plate is followed by a tube of the same size, and whose length is from 2 to 3 times its diameter, there will be no contraction at the end ($\mu = 1$), but the velocity will be but $0.82\sqrt{2\ g\ h}$, hence $n = 0.82$; $m = \mu\ n = 0.82$. Or

$$v = 0.82\sqrt{2\ g\left(z + \frac{p - p'}{\gamma}\right)}$$

$$Q = 0.82\ a\sqrt{2\ g\left(z + \frac{p - p'}{\gamma}\right)}$$

$$\frac{v^2}{2\ g} = \frac{2}{3}\left(z + \frac{p - p'}{\gamma}\right) \quad \therefore$$

$$\theta = \text{loss of head} = \tfrac{1}{3}\left(z + \frac{p - p'}{\gamma}\right) = \tfrac{1}{2}\ \frac{v^2}{2\ g}.$$

These facts, found by experiment, are accounted for as follows (fig. 28): the vein contracts from A B, the edge of the orifice, to $a\ b$, the contracted vein, beyond which it expands and fills the tube, flowing full at C D if the tube is not too short. Experiment has shown the proper length to be from 2 to 3 times the diameter. If the length is less than this, the discharge takes place as through a thin plate, the vein not touching the tube: if it be greater, there is a loss by friction on the sides of the tube, and the discharge is diminished. In the case supposed, then, there

is a loss of head due to the enlargement of the vein beyond $c\ d$, from the area $m\ a$ to the area a. Hence

$$\theta = \frac{Q^2}{2\ g\ a^2}\left(\frac{1}{m} - 1\right)^2 = \frac{v^2}{2g}\left(\frac{1}{m} - 1\right)^2$$

and we therefore have

$$\frac{v^2}{2\,g} = z + \frac{p - p'}{\gamma} - \frac{v^2}{2\,g}\left(\frac{1}{m} - 1\right)^2$$

$$= \frac{z + \frac{p - p'}{\gamma}}{1 + \left(\frac{1}{m} - 1\right)^2}$$

If we suppose $m = 0.62$, we find from the above formula

$$\theta = 0.273\left(z + \frac{p - p'}{\gamma}\right)$$

while, as we have seen, experiment shows the co-efficient to be 0.33. To be more accurate, experiment shows that

$$\theta = 0.487\ \frac{v^2}{2g}\ .$$

while, theoretically, $\theta = \left(\frac{1}{m} - 1\right)\frac{v^2}{2g}$.

The value of m which would make these two equations agree, is $m = 0.59$, which is not far from its real value, and sufficiently close to show that our theory accounts for the phenomena.

The effect of the mouthpiece is, therefore, to increase Q and to diminish v. Now, the only way in which (the contraction remaining the same) the discharge from a given orifice can be augmented, is by an increase in the pressure p in the reservoir, or a decrease in the pressure on the contracted vein. In the present case it must be the latter reason. Let us find the pressure p'' on the contracted vein, by applying the theorem of Bernouilli between the reservoir and the contracted vein; then

$$z_1 + \frac{p}{\gamma} = z_2 + \frac{p''}{\gamma} + \frac{v''^2}{2\,g}$$

$$= z_2 + \frac{p''}{\gamma} + \frac{Q^2}{2\,g\,m^2\,a^2}$$

$$\frac{p - p''}{\gamma} + z = \frac{Q^2}{2\,g\,m^2\,a^2}$$

$$Q^2 = 2\,g\,m^2\,a^2\left(z + \frac{p - p''}{\gamma}\right)$$

But we have found by experiment

$$Q^2 = (0.82)^2\ a^2 .\ 2g\left(z + \frac{p-p'}{\gamma}\right)$$

Equating these two values of Q^2, we have, calling $0.82 = m_1$,

$$m^2\left(z + \frac{p-p''}{\gamma}\right) = m_1^2\left(z + \frac{p-p'}{\gamma}\right)$$

from which

$$\frac{p'}{\gamma} - \frac{p''}{\gamma} = \left(\frac{m_1^2}{m^2} - 1\right)\left(z + \frac{p-p'}{\gamma}\right)$$

As $m_1 > m$ this is always positive, hence $p' > p''$. Replacing the values of m and m_1 (0.62 and 0.82), we find

$$\frac{p'-p''}{\gamma} = \tfrac{3}{4}\left(z + \frac{p-p'}{\gamma}\right)$$

Hence the height due to the pressure on the contracted vein should be less than that due to the pressure at C D, or p', by three-fourths of the head $\left(z + \frac{p-p'}{\gamma}\right)$.

Venturi confirmed these results by some remarkable experiments (fig. 79). He first found by experiment that $n = 0.822$. He then bored twelve small holes in the mouthpiece, around the contracted vein, the result of which was that C D did not run full, and Q was exactly the same as if there had been no mouthpiece, while not a drop of water ran through the small holes. He then inserted a glass siphon into the contracted vein, the small holes having all been closed. The siphon dipped into colored water at its lower end, and the value of $z + \frac{p-p'}{\gamma}$ was $0.88^{m.}$ As soon as the discharge was allowed to take place, the colored water rose in the siphon to a height of $0.65^{m.}$, which is almost exactly three-fourths of $0.86^{m.}$. If we had taken $m = 0.62$, $m_1 = 0.8175$, the result of the experiment would agree exactly with the theoretical result.

We have hitherto compared theory with experiment only in cases where the discharge occurred through a thin plate, where m was about 0.62. In any other case our general equation will give correct results, if we substitute the proper value of m. Thus, Bidone found from experiments with a re-entering mouthpiece that the thickness of the mouthpiece had a considerable effect upon the discharge, as shown in the following table, where r = radius of cylindrical mouthpiece, and e = its thickness. It is to be remembered that the tube could be made to flow full, like a long outside mouthpiece, or could be allowed to flow as an orifice, without the water touching the sides of the tube: —

Tube not Filled.		Tube Filled.	
Minimum value of e, almost 0.	$e > 0.414\ r.$	e almost 0.	$e > 0.414\ r.$
$m = 0.50.$	$m = 0.61$	$m = 0.7071.$	$m = 0.8125.$

These results are very easily explained with the tube not filled; a thin tube gave the theoretical co-efficient 0.50; while with a thick tube the condition of things was more like an orifice in a thin plate forming the side of the vessel, and the co-efficient was 0.61. With the tube filled, the value of m should be theoretically

$$\sqrt{\frac{1}{1+\left(\frac{1}{m}-1\right)^2}},$$

according to the equation on page 41, in which m is supposed to be the co-efficient if there is no mouthpiece, or if it does not flow full. Hence for a thin tube $m = 0.50$ in the above value, and the resultant co-efficient is 0.707, exactly as given in the table. For a thick tube, supposing the co-efficient with the tube not filled to be 0.61, the value of the co-efficient in the last column should be 0.843, which agrees tolerably well with the result in the table.

The mouthpiece just discussed, having the effect of increasing Q at the expense of v, and therefore decreasing the energy of the issuing liquid, should be used only when the object is to augment the discharge, and not when it is desirable to obtain as much work as possible from the water.

We have found that

$$\frac{p''}{\gamma} = \frac{p'}{\gamma} - \left(\frac{m_1^2}{m^2} - 1\right)\left(z + \frac{p - p'}{\gamma}\right)$$

In order that p'' shall be always greater than 0, it is necessary that

$$\frac{p'}{\gamma} > \left(\frac{m_1^2}{m^2} - 1\right)\left(z + \frac{p - p'}{\gamma}\right) \quad \text{or}$$

$$\frac{p'}{\gamma} > \cdot\ \tfrac{3}{4}\left(z + \frac{p - p'}{\gamma}\right), \text{ or, if } p = p'$$

$$\frac{p'}{\gamma} > \tfrac{3}{4}\ z\ ;\ z < \frac{4}{3}\ \frac{p'}{\gamma}, \quad \text{or}$$

$$z < \frac{4}{3}\ 34'$$

$$z < 45'\ .$$

Hence if z is greater than about 45 feet, the discharge cannot take place as supposed, at least, not with the assumed values of m_1 and m.

CONICAL MOUTHPIECE. — Such mouthpieces may be *convergent* or *divergent.*

(a.) *Convergent mouthpiece.* — Up to a certain limit, the effect of convergent mouthpieces is to increase Q still more than in the case of cylindrical mouthpieces; for μ still remains sensibly equal to unity, while there is less loss of head. The co-efficient depends upon the angle of convergence, or the angle at the apex of the cone, a; and it appears from experiments of d'Aubuisson and Castel that as a is increased, m increases, reaching its maximum at 13° or 14°, then decreasing to the ordinary value of about 0.61 for $a = 18°$; while the co-efficient of velocity increased regularly with a, from 0.82 to 0.98 when $a = 180°$.

Divergent mouthpiece. — Let fig. 40 represent a mouthpiece composed of the part A B C D, having the shape of the contracted vein, to which is fitted tangentially a conical divergent mouthpiece C D E F, the curves being so smooth and gradual that no loss of head occurs anywhere. Then, supposing the mouthpiece to flow full, we find the velocity v at E F to be

$$v = \sqrt{2\,g\left(z + \frac{p - p'}{\gamma}\right)} \qquad \text{and}$$

$$Q = S\sqrt{2\,g\left(z + \frac{p - p'}{\gamma}\right)}$$

if S is the area of E F. It would seem, then, at first sight that by increasing S we could increase Q as much as we pleased; but, as we have seen, such increase can only be due to a corresponding decrease of pressure on the contracted vein C D. Let p'' be the pressure on the contracted vein, and v'' the velocity through it; then

$$v'' = \sqrt{2\,g\left(z + \frac{p - p''}{\gamma}\right)}$$

and, if A is the area of the contracted vein,

$$Q = A\sqrt{2\,g\left(z + \frac{p - p''}{\gamma}\right)}$$

Equating the two values of Q.

$$\frac{S}{A} = \sqrt{\frac{z + \frac{p-p''}{\gamma}}{z + \frac{p-p'}{\gamma}}}.$$

and the maximum possible value of S will occur when $p'' = 0$, or

$$\text{max. } \frac{S}{A} = \sqrt{\frac{z + \frac{p}{\gamma}}{z + \frac{p-p'}{\gamma}}}$$

If $p = p'$, we have

$$\text{max. } \frac{S}{A} = \sqrt{\frac{z + \frac{p}{\gamma}}{z}} = \sqrt{1 + \frac{p}{\gamma z}}$$

$\text{max. } Q = A\sqrt{2g\left(z + \frac{p}{\gamma}\right)}$ which is the same as the discharge into a vacuum.

This effect of a diverging tube may be entirely suppressed by boring a few capillary holes around the contracted vein, as Venturi found by experiment.

Submerged diverging tube. — Let fig. 41 represent a diverging tube whose axis is at a depth h_2 below the lower water. Then if S and v refer to E F, and A, v_o'' and p'' refer to C D; p being the pressure on the surface in the upper reservoir, and p' on the lower reservoir; we have

$$v = \sqrt{2g\left(H + \frac{p-p'}{\gamma}\right)}$$

$$Q = S\sqrt{2g\left(H + \frac{p-p'}{\gamma}\right)}$$

$$Q = A\sqrt{2g\left(h_1 + \frac{p-p''}{\gamma}\right)} \quad \therefore$$

$$\frac{S}{A} = \sqrt{\frac{h_1 + \frac{p-p''}{\gamma}}{H + \frac{p-p'}{\gamma}}}$$

$$\text{max. } \frac{S}{A} = \sqrt{\frac{h_1 + \frac{p}{\gamma}}{H + \frac{p - p'}{\gamma}}} .$$

$$\text{If } p - p' : \qquad \text{max. } \frac{S}{A} = \sqrt{\frac{h_1 + \frac{p}{\gamma}}{H}}$$

$$\text{Max. } Q = A \sqrt{2g \left(h_1 + \frac{p}{\gamma} \right)}$$

These principles find their application in the so-called *diffuser*, used with some turbine wheels.

We have thus far considered that the divergent mouthpiece is so shaped as to avoid all loss of head. Should it, however, be a simple conical tube, there would be a loss of head at the entrance, and this loss would be greater than in the case of a cylindrical mouthpiece. Experiments on this point are not numerous, and the co-efficients are not well known.

TIME REQUIRED TO EMPTY VESSELS BY DISCHARGE THROUGH AN ORIFICE. — We have hitherto considered only cases of discharge under a constant head, or where the level of the water was kept constant in the reservoir. We have now to consider a case where that level is allowed to sink according to the quantity of water discharged, being unsupplied. Let it be required to find the time necessary to lower the water level in a vertical prismatic vessel of area A from a height h_1 to a height h_2 above its horizontal base, the water being discharged through an orifice of area a in that base (fig. 82). In a time $d\,t$ the discharge will evidently be, if the head is x,

$$m\,a\,d\,t\sqrt{2\,g\,x} = -\,A\,d\,x \qquad \text{or}$$

$$d\,t = -\,\frac{A}{m\,a\sqrt{2\,g}}\,x^{-\frac{1}{2}}\,d\,x \quad \therefore$$

$$t = -\,\frac{A}{m\,a\sqrt{2\,g}} \int_{h_1}^{h_2} x^{-\frac{1}{2}}\,d\,x$$

$$= \frac{2\,A}{m\,a\sqrt{2\,g}} \left(\sqrt{h_1} - \sqrt{h_2} \right)$$

For $h_2 = 0$, we find the time necessary to empty the vessel from a height h_1 to be

$$t_o = \frac{2\ A}{m\ a\ \sqrt{2\ g}}\ \sqrt{h_1}$$

The quantity which has been discharged is A h. Had the water been kept at a constant height h, the time necessary to discharge this quantity would be

$$t = \frac{A\ h}{m\ a\ \sqrt{2\ g\ h}} = \frac{A}{m\ a\ \sqrt{2\ g}}\ \sqrt{h}\ .$$

Hence *the time necessary to empty a cylindrical vessel is just double that necessary to discharge the same quantity under the constant head h.* This theorem is not exactly true, for we have supposed the discharge to take place to the very last with the orifice running full. But, as is well known, before the water is entirely discharged, a funnel will be formed over the orifice, and the law of discharge somewhat interrupted, the time being increased to some extent.

Discharge over weirs (Fig. 43). — A weir is an orifice uncovered at the top. The formulæ for orifices may, therefore, be applied to weirs by making H = 0. Weirs are generally rectangular, with a horizontal crest or sill, and vertical sides, in which case they are commonly called notches. The contraction on a weir takes place, as in an orifice, on all four sides, when the edges of the orifice are sharp; for it is found that the surface falls for some distance back from the weir, so that the depth of water d actually passing the weir is less than the depth h of the sill below the water level some distance back. Let l = length of weir, h this depth, and Q the discharge, then from the formulæ for orifices we shall have the following formulæ for this case: —

(A.) Not taking account of the velocity of approach.

(1) Approximate: $Q = m\ A\ \sqrt{2\ g\ \frac{h}{2}}$

$$= \frac{m\ l\ h}{\sqrt{2}}\ \sqrt{2\ g\ h}$$

$$= 0.707\ m\ l\ h\ \sqrt{2\ g\ h}\ .$$

(2) By integrating $Q = 0.667\ m\ l\ h\ \sqrt{2\ g\ h}\ .$

(B.) Taking account of velocity of approach.

(3) Approximate: $Q = m_1 A \sqrt{\frac{2 g \frac{h}{2}}{1 - \frac{A^2}{A_1^2}}}$

in which A_1 is the area of channel of approach, and m_1 is co-efficient of discharge. We have seen that the velocity of approach affects the co-efficient, so that m_1 is rather larger than m.

The formula (3) may be written

(4) $$Q = \frac{0.707\ m_1\ l\ h}{\sqrt{1 - \frac{A^2}{A_1^2}}} \sqrt{2\ g\ h} \quad \text{(approx.)}$$

Or, if we wish this in another form, we have

(5) $$Q = m_1\ l\ h \sqrt{2g \left(\frac{h}{2} + \frac{v_o^2}{2g} \right)} \quad \text{(approx.)}$$

v_o being the velocity of approach. More exactly, by integrating, we shall have

(6) $$Q = 0.667\ m_1\ l \sqrt{2\ g} \left[\left(h + \frac{v_o^2}{2\ g} \right)^{\frac{3}{2}} - \left(\frac{v_1^2}{2\ g} \right)^{\frac{3}{2}} \right]$$

If, in equations (1) and (2), we substitute $m = 0.61$, we shall have

(7) $$Q = 0.43\ l\ h \sqrt{2\ g\ h} = 3.45\ l\ h^{\frac{3}{2}} \quad \text{(approx.)}$$

(8) $$Q = 0.41\ l\ h \sqrt{2\ g\ h} = 3.29\ l\ h^{\frac{3}{2}}$$

Experiments have generally, however, determined directly the value of the co-efficient $\frac{2}{3}\ m = c$, so that

(9) $$Q = c\ l\ h \sqrt{2\ g\ h} = c'\ l\ h^{\frac{3}{2}}$$

is the general formula used for calculating the discharge over rectangular weirs. *Lesbros* found c from *0.371* to *0.424* (c' from 2.975 to 3.40) for weirs 8 inches long, with complete contraction, and where l was less than one-tenth B, but greater than three and a quarter inches, B being the width of the channel of approach. For weirs without end contraction, Lesbros found $c = 0.45$ ($c' = 3.61$) (average).

In order that contraction may be complete, the up-stream face of the weir must be plane and vertical, and the edges of the crest and sides sharp and thin, or else bevelled off down stream. The edges must not be

rounded or bevelled on the up-stream side to any extent. Lesbros found that the effect of thick edges and crest was to increase the value of c. In order that contraction may be complete, it is further essential that the crest and sides of the weir should be sufficiently removed from the bottom and sides of the channel of approach respectively. If a weir is set in a channel so as to extend completely across it, there will be no end contraction and c will be correspondingly greater. Castel found that as the ratio $\frac{l}{B}$ increased, so did also the value of c, according to the following table: —

$\frac{l}{B}$	1.0	0.90	0.80	0.70	0.60	0.50	0.40	0.30	0.25
c	0.443	0.438	0.431	0.423	0.416	0.410	0.405	0.399	0.397

These results may be expressed by the following formula: —

$$Q = \left(0.381 + 0.062 \frac{l}{B}\right) l h \sqrt{2 g h} = 3.05 \left(1 + 0.163 \frac{l}{B}\right) l h^{\frac{3}{2}}$$

For $l = B$; this becomes

$$Q = (0.381 + 0.062)\, l h \sqrt{2 g h}$$

$$= 0.443\, l h \sqrt{2 g h} \cdot = 3.55\, l h^{\frac{3}{2}}$$

In order that this equation should be applicable, the following conditions must be satisfied: —

1°. The weir must have a sharp, horizontal crest.

2°. The crest must be at least 2 h above the water below the weir.

3°. $l h$ must be less than $\frac{1}{3}$ the area of the channel of approach.

4°. l must be at least equal to $\frac{1}{3}$ B.

J. B. Francis, of Lowell, has given a weir formula, which is better suited for practical use than any other, though not applicable in all cases. He considers the contraction at the sides independent of l, at least when l is greater than a certain limit, and he finds that contraction to be proportional to h. At each end of the weir, therefore, where there is complete contraction, he considers the effective length of the weir diminished by (0.10 h). His formula is, therefore,

$$Q = 3.33\, (l - 0.1\, n\, h)\, h^{\frac{3}{2}}.$$

in which n is the number of end contractions. In an ordinary weir, with perfect end contraction, $n = 2$; if the weir has a length equal to B, then $n = 0$; and if the weir be divided into two parts by a vertical pier or partition, then each part may be considered separately, or the two may be

taken together, by making $n = 4$. The constants for this formula were obtained from experiments on weirs ten feet long, with heads up to 19″, while all previous experiments had been made on weirs of very small dimensions. His formula is only applicable under certain conditions, viz. : —

(1.) It applies to rectangular weirs in the side of a dam which is vertical on the up-stream side; the crest being horizontal, sides vertical, and edges sharp, so that the vein will not touch the sides of the weir after once passing its up-stream edge.

(2.) It is only applicable when $l > 3\ h$, and it is not applicable to very small heads; the limits of h in the experiments were from 7″ to 19″, but the equation would probably apply from $h = 6'$ to $h = 24''$.

(3.) The end contraction must be either complete or entirely suppressed. The least distance from the end of the weir to the side of the canal is h; the least depth of the channel of approach is $3\ h$.

(4.) The form of the channel of approach should be such that the velocity over all parts of the weir should be the same, and if the water reaches the channel in a turbulent condition, gratings, or wooden dams perforated with a number of one-inch holes, should be placed in the channel, as far from the weir as possible, to calm the water before it approaches the weir.

(5.) The under side of the vein, after passing the weir, should have free communication with the atmosphere, the water below the weir should not be higher than $\frac{1}{2}\ h$ below the crest of the weir, and should be lower still when l is large. Mr. Francis' experiments showed that by raising the water, under certain circumstances, an increased discharge would result, owing to the formation of a partial vacuum under the vein on the down-stream side. With no end contraction, this formula becomes

$$Q = 3.33\ 1\ h_2^{\frac{3}{2}}$$

Tables may be calculated giving according to this formula the values of Q for any h. Probably the best method of studying weir formulæ is to first find a formula like the above, applicable to a sharp-crested weir, with no end contraction, and with no appreciable velocity of approach, and then to determine the corrections to be made for other conditions. This method was followed by Fteley and Stearns, in their paper giving an account of their experiments at Framingham (Trans. A. S. C. E., 1883).

These experimenters found for a weir with no end contraction and no velocity of approach the general formula

$$Q = 3.31\ l\ h_2^{\frac{3}{2}} + 0.007\ l\ .$$

This applies where h is not less than 0.07 feet.

Correction for end contraction. —We have seen that Francis diminishes the length of the weir by $0.1\ h$ for each end contraction. Fteley and

Stearns' experiments showed that the diminution was by no means proportional to h, but was very irregular. The amount of diminution will evidently vary with the velocity of approach, for we have seen that such velocity always affects the co-efficient of contraction.

Correction for velocity of approach. — There are three methods of correcting for velocity of approach: —

1°. By substituting for $h_1^{\frac{3}{2}}$ in the formula the quantity

$$\left(h + \frac{v_o^2}{2g}\right)^{\frac{3}{2}} - \left(\frac{v_o^2}{3g}\right)^{\frac{3}{2}}.$$

according to the formula already given,

$$Q = \tfrac{2}{3}\, m_1\, l \sqrt{2g} \left[\left(h + \frac{v_o^2}{2g}\right)^{\frac{3}{2}} - \left(\frac{v_o^2}{2g}\right)^{\frac{3}{2}}\right]$$

This method is followed by Mr. Francis, whose formula becomes

$$Q = 3.33\ (l - 0.1\ n\ h) \left[\left(h + \frac{v_o^2}{2g}\right)^{\frac{3}{2}} - \left(\frac{v_o^2}{2g}\right)^{\frac{3}{2}}\right]$$

This cannot be exact, as it ignores the effect of the velocity of approach on the co-efficient. In the application of this method v_o is first to be supposed zero, and an approximate Q calculated, from which a close value of v_o is obtained. The above formula is then applied, and a new value of Q obtained. If necessary, the operation may be repeated a sufficient number of times to give an accurate result.

2.° By multiplying the quantity given by the formula

$$Q = 3.33\ (l - 0.1\ n\ h)\ h_2^{\frac{3}{2}},$$

h being the observed depth on the weir, by a constant depending upon the ratio of $(l\ h)$ to (B T.). Hunking and Hart have found that results identical with these obtained by the method just explained may be obtained by multiplying Q by the quantity

$$k = 1 + 0.2489\ \frac{h\ (l - 0.1\ n\ h)}{\text{B T.}}$$

and they give a table of values of k which saves time in computation. The use of this method, giving results identical with that just described, is affected by the same errors. To be correct, the co-efficient should be larger the larger the velocity of approach. Weisbach corrects for velocity of approach in a similar manner. He gives the following general formulæ for weirs: —

If $l < B$:

$$Q = \tfrac{2}{3} m \left[1 + 1.718 \left(\frac{l h}{B T} \right)^4 \right] l h \sqrt{2 g h}$$

If $l = B$:

$$Q = \tfrac{2}{3} m \left[1.041 + 0.3693 \left(\frac{h}{T} \right)^2 \right] l h \sqrt{2 g h}$$

In these equations B and T are the breadth and depth respectively of the channel of approach, and m is the co-efficient derived from Lesbros' experiments on small orifices, varying from 0.371 to 0.424. The fact that this formula depends upon Lesbros' experiments, which were made on a small scale, renders it of little use in practice.

(3°.) The third method of taking account of velocity of approach is to correct the observed depth on the weir, h, by adding the quantity C, before applying the formula for weir discharge. Fteley and Stearns have found the value of this correction for various cases; they should only be used, however, when the weir formula of those authors is also used. When Francis' formula for weirs is used, his method of correcting should be used also, as his experiments and calculations were made in this way. Fteley and Stearns found

$C = 1.5\ h$ for weirs without end contraction.

$C = 2.05\ h$ for weirs with end contraction.

Effect of a wide crest. — All the formulæ thus far given suppose a sharp crest which the liquid vein does not touch after leaving its up-stream edge. Fteley and Stearns have experimented on the effect of a wide crest, using crests from two to ten inches in width, and with values of h from 0.12 to 0.89 feet. They found that for any given crest, of width w, there was a certain value of h at which no correction was required, the flow being the same as with a sharp crest and the same h. This value of h, which we may call h^1, was

$$h^1 = 1.614\ w.$$

If h was below this value, a certain correction C must be subtracted from the depth on the wide crest to obtain the equivalent depth on a sharp crest. If h is above the value, a correction C must be added to the length on the wide crest. The correction is given by the formula

$$C = 0.2016 \sqrt{y^2 + 0.2146\ w^2} - 0.1876\ w$$

where $y = 0.807\ w - h$.

They give a table, from which this correction may be obtained with ease.

Effect of a rounded crest.—We have hitherto supposed the up-stream edge of the weir to be sharp. If it is a quarter circle with a radius R, Fteley and Stearns have found that the value of h should be increased by adding $C = 0.7\ R$. This is only applicable, however, when $R > \frac{1}{2}''$, and when h is great enough so that the vein is raised from the crest, as in a sharp-crested weir. Experiments were made on a weir with both wide and rounded crest, in order to see whether each could be corrected for as above, but it was found that the correction for the rounded edge was 0.41 R. This value is limited to cases where R is less than $\frac{1}{2}''$, and h is less than 0.17′ and 0.26′ for radii of $\frac{1}{4}$ and $\frac{1}{2}$ inch respectively, and $w = 4''$ to $5.''$

Other formulæ for weirs.

1°. *Boileau.*—Weirs without end contractions.

$$Q = \frac{\sqrt{1 - \frac{d}{h}}}{\sqrt{1 - \left(\frac{h}{h+s}\right)^2}}\ l\ h \sqrt{2\,g\,h}.$$

h = depth on weir measured far enough back to be beyond where curvature of sheet commences.

d = actual depth over crest.

s = height of crest above bottom.

This formula is of little practical value now.

2°. *Braschmann.*

$$Q = \left[0.3838 + 0.0386\ \frac{l}{B} + \frac{0.00053}{h}\right] l\ h \sqrt{2\,g\,h}.$$

If $l = B$

$$Q = \left[0.4224 + \frac{0.00053}{h}\right] l\ h \sqrt{2\,g\,h}.$$

3°. *Bornemann.*—Weirs with no end contraction.

$$Q = \left(0.5673 - 0.1239 \sqrt{\frac{h}{T}}\right) l\ h \sqrt{2\,g\,h}$$

$$\text{if } h < \tfrac{1}{3}\,T.$$

$$Q = \left(0.6402 - 0.2862 \sqrt{\frac{h}{T}}\right) l\ (h + h_1) \sqrt{2\,g\,(h + h_1)}$$

$$\text{if } h > \tfrac{1}{3}\,T.$$

in which $h_1 = \frac{v_o^2}{2\,g}$.

Oblique weirs.—Weirs oblique to the axis of the channel are generally calculated as though they were weirs of equal length at right angles to channel, although in reality the quantity discharged by them is smaller. Boileau found where the weir made angles of 45° and 63° 25′ with the axis of the channel the quantity discharged was, respectively, 0.911, and 0.942 of that given by the usual formula applied to the same length of weir.

Experiments on weirs.—There are two methods of experimenting upon flow over weirs. *First*, by measuring directly the quantity flowing in a given time, varying the conditions according to the object to be attained. This method involves an accurate measurement of quantity, of time, and the beginning and end of the experiment must be accurately controlled, so that the method is difficult and expensive with large quantities. *Second*, a constant quantity of water may be allowed to flow over weirs under different conditions, and from the varying depths on the weir the effect of those conditions may be determined. Thus, if the same quantity be allowed to flow over a sharp-crested weir, and then over a wide-crested weir, the change in depth will show the effect of the wide crest. This method dispenses with a direct measurement of quantity, but does not allow of a complete solution of all questions relating to weirs. To show its application to the determination of constants, suppose a weir without end contraction, and let the formula for flow be assumed of the form $Q = c\, l\, h^x$, in which c and x are unknown. Then, if we allow the same Q to flow over two weirs of lengths l and l', at depths h and h', we have

$$Q = c\, l\, h^x = c\, l'\, h'^x \quad \therefore$$

$$x = \frac{\log.\ l' - \log.\ l}{\log.\ h - \log.\ h'}$$

We may thus, by two experiments, find the power of h which will give the same Q in both cases. In order to find c, a direct measurement of Q would be necessary. Fteley and Stearns used the second method in studying the effect of velocity of approach, end contraction, wide and rounded crests.

The experiments on weirs, previous to those of Mr. Francis, were made on such a small scale that the results are of little value. The best experiments of late years are those of Francis, and of Fteley and Stearns. Mr. Francis experimented with a weir 10 feet long, with h from 7 to 19 inches, the measuring vessel being a lock chamber with a capacity of 12138 cubic feet. Fteley and Stearns used a weir 5 feet long, and one 19 feet long, with three measuring vessels, the smallest with a capacity of 359 cubic feet, the largest with over 300,000 cubic feet. The results of these experiments have been already given.

Effect of height of water on down-stream side.—For the application of the weir formulæ which have been given, the under side of the weir should have free communication with the atmosphere. Mr. Francis found that by raising the water on the down-stream side, the discharge could sometimes be increased, due to the exhaustion of the air from beneath the surface of the sheet, where there is no end contraction, and the sheet is prevented from expanding laterally after leaving the crest. When h was 0.85′ no effect was observed when the water on the down-stream side was 0.235′ below crest; when level with the crest, the effect was very small; when $\frac{3}{4}''$ above the crest, the discharge was increased by 0.7 of one per cent, and when 1.25″ above the crest the discharge was decreased, the change being rapid for greater heights, the weir being then submerged. It is best to keep the lower water at least $\frac{1}{2}\ h$ below the crest of the weir, when the under side of the sheet communicates freely with the atmosphere at the ends. Fteley and Stearns found that if h is considerable, the lower water could rise to the level of the crest of the weir without affecting the discharge, and that the error would not be over one per cent, if the lower water should rise to a height of .15 h above the crest.

Measurement of h. — The value of h, to be used in the formulæ given, denotes the height above the crest of the weir of the water in the still pond, or back of the point where the curvature of the surface commences. Mr. Mills has shown that in order that a peizometric column shall indicate truly the height of the surface of water in motion, the velocity should be neither increasing or decreasing, the currents flowing parallel to the sides of the conduit, and the orifices of the peizometers should have edges parallel to the direction of the current, or in the plane of the side of the conduit, and passages normal to that direction. These conditions being fulfilled, the value of h should be measured at a distance equal to 2.5 S back of the weir, S being the height of the crest above the bottom of the channel. The channel of approach should be of uniform section. The measurement of h may be made with a hook gauge. The experiments of Francis, as well as those of Fteley and Stearns, show that there is a triangular space between the weir and the bottom of the channel, extending back for a distance of about $2\frac{1}{2}$ S, within which the pressure is greater than farther from the weir, although the amount of the excess of pressure is quite uncertain.

Triangular notch. — The formulæ for triangular orifice give for a notch,

$$Q = \frac{4}{15}\, m \sqrt{2\,g}\, .\, \frac{l}{h}\, .\, h^{\frac{5}{2}}$$

The constants for this formula have not been much studied. We may, however, give to m the same value that it has in Francis' formula for weirs. This gives us for the notch

$$Q = 1.333 \frac{l}{h} \cdot h^{\frac{5}{2}}$$

$$= 2.6667 \text{ cot. } a \cdot h^{\frac{5}{2}}$$

if a is the angle the edge of the notch makes with the horizontal. From this we may find a formula for a weir whose crest is not level. Let Q_1, Q_2, Q_3, be respectively the quantities flowing through the areas A D B, A F G, and F G D B, in fig. 44. Then if B D $= h$; F G $= h_2$; F B $= l_2$

$$Q_1 = 1.333 \text{ cot. } a \cdot h^{\frac{5}{2}}$$

$$Q_2 = 1.333 \text{ cot. } a \, h_2^{\frac{5}{2}}$$

$$Q_3 = Q_1 - Q_2 = 1.333 \text{ cot. } a \left[h^{\frac{5}{2}} - h_2^{\frac{5}{2}} \right]$$

$$\text{cot. } a = \frac{l_2}{h - h_2}$$

$$Q_3 = 1.333 \frac{l_2}{h - h_2} \left[h_2^{\frac{5}{2}} - h_2^{\frac{5}{2}} \right]$$

Correcting for end contractions, this formula becomes

$$Q^3 = 1.333 \frac{l_2 - 0.1\, n \frac{h + h_2}{2}}{h - h_2} \left(h^{\frac{5}{2}} - h_2^{\frac{5}{2}} \right)$$

a formula applicable under the same limitations as Francis' weir formula. This formula is very useful where the crest of a weir is not exactly horizontal.

Depth on crest of weir. — Fteley and Stearns found the actual depth on the crest of the weir to vary from 0.852 h to 0.882 h when the velocity of approach varied between 0.389 and 1.529 feet per second.

Submerged weirs (Fig 45).—A weir is submerged if the lower water rises above its crest, as in fig. 45. We have seen that if H''' is small, the effect of the submergence is inappreciable. If H''' is large, the usual formula is obtained by considering the discharge in two parts, and is

$$Q = c. \tfrac{2}{3} l \sqrt{2g} \, (H'' - H''')^{\frac{3}{2}} + c' \, l \, H''' \sqrt{2g \, (H'' - H''')}$$

Mr. Francis finds from his experiments

$$Q = 3.33\ l\ (H'' - H''')^{\frac{3}{2}} + 4.5988\ l\ H'''\sqrt{H'' - H'''}$$

$$= 3.33\ l\ \sqrt{H'' - H'''}\ (H'' + 0.381\ H''')$$

or, if there is end contraction,

$$Q = 3.33\ (l - 0.1\ n\ H'')\sqrt{H'' - H'''}\ (H'' + 0.381\ H''')$$

Fteley and Stearns propose the formula

$$Q = c\ l\left(H'' + \frac{H'''}{2}\right)\sqrt{H'' - H'''}$$

for weirs without end contraction, and with no velocity of approach. They found c to vary with the ratio of H''' to H'', and gave a table from which its value could be taken at once. They consider that although in their experiments the value of H'' only varied from 0.3251′ to 0.8149′, the formula will apply to much greater depths. For values of $\frac{H''}{H'''}$ less than 0.08 the formula is not applicable. Correction may be made for end contractions in the usual way; also for velocity of approach; but the formula should not be used if the velocity of approach, or that with which the water leaves the weir, is large. The channel, on the down-stream side, should be sufficiently deep and wide to make the velocity small, and this precaution is more important as the weir becomes more submerged.

Lesbros gave the formula

$$Q = m\ l\ H''\sqrt{2\ g\ (H'' - H''')}$$

$$= m\sqrt{2\ g}\ l H''\sqrt{H'' - H'''}$$

His experiments were made with a weir 0.24^{m} long, and with end contraction. The following are his values of m: —

$\frac{H''-H'''}{H''}$	0.002	.003	.004	.005	.006	.007	.008	.009	.010	.015
m	.295	.363	.430	.496	.556	.597	.605	.600	.596	.580
$\frac{H''-H'''}{H''}$	.020	.025	.030	.035	.040	.045	.050	.060	.080	0.10
m	.570	.557	.546	.537	.531	.523	.522	.519	.517	.516
$\frac{H''-H'''}{H''}$	0.15	.20	.25	.30	.35	.40	.45	.50	.55	.60
m	.512	.507	.502	.497	.492	.487	.480	.474	.466	.459
$\frac{H''-H'''}{H''}$	0.70	0.80	0.90	1.0						
m	.444	.427	.409	.390						

The most general formula for a submerged weir, taking account of velocity of approach, would be

$$Q = l\sqrt{2g}\left\{\tfrac{2}{3}c'\left[\left(H'' - H''' + \frac{v_o^2}{2g}\right)^{\frac{3}{2}} - \left(\frac{v_o^2}{2g}\right)^{\frac{3}{2}}\right] + c''H'''\left(H'' - H''' + \frac{v_o^2}{2g}\right)^{\frac{1}{2}}\right]$$

Bornemann finds in this formula

$$c' = c'' = 0.702 - 0.2226\sqrt{\frac{H'' - H'''}{l}} + 0.1845\left(\frac{H'''}{H''}\right)^2$$

CHAPTER III.

Flow of Water in Open Channels.

Definitions.— Considering a portion of the stream which is approximately straight, take a section at right angles to the stream, and we let

F = area of water section.

p = wetted perimeter.

b = width at surface of water.

$t_m = \frac{F}{b}$ = mean depth.

$R = \frac{F}{p}$ = hydraulic mean depth.

(a) Rectangular section (Fig. 46).

$$F = b\,d$$

$$p = b + 2\,d$$

$$R = \frac{F}{p} = \frac{b\,d}{b + 2\,d} = \frac{d}{1 + \frac{2\,d}{b}}$$

$$t_m = d\,.$$

If $\frac{d}{b}$ is very small, $R = d = t_m = \frac{F}{b}$

(b) Circular section. Radius r.

$$F = \pi r^2 \; ; \; p = 2 \pi r \text{ (if flowing full).}$$

$$R = \frac{F}{p} = \frac{r}{2} .$$

The value of R is the same whether the section be full or half full.

(c) Trapezoidal section (Fig. 47).

$$F = T \frac{B_1 + B_o}{2}$$

$$= B_o T + T^2 \cot. a$$

$$= T (B_o + T \cot. a)$$

$$= T (B_1 - T \cot. a)$$

$$p = B_o + \frac{2 T}{\sin. a}$$

$$R = \frac{F}{p} = \frac{T (B_o + T \cot. a)}{B_o + \frac{2 T}{\sin. a}} .$$

To find the values of B_o and B_1 with a given F and T, we have

$$B_o = \frac{F}{T} - T \cot. a$$

$$B_1 = \frac{F}{T} + T \cot. a$$

Hence, $$p = \frac{F}{T} + \frac{T}{\sin. a} (2 - \cos. a)$$

$$\frac{p}{F} = \frac{1}{R} = \frac{1}{T} + \frac{T (2 - \cos. a)}{F \sin. a} .$$

(d) Irregular section. Find F by Simpson's rule and measure p. For ordinary streams, $p = b$ very nearly, or very accurately $p = 1.01\ b$.

Form of section for maximum R. — We shall see shortly that, in some respects, the section which has the maximum value of R is in some respects the most favorable. Assuming a constant area F, we find the proper proportions as follows: —

(a) *Rectangular section.* Call $b\ d = F = c$.

Then $$R = \frac{c}{\frac{c}{d} + 2\ d}$$

If R is a maximum $\frac{c}{d} = 2\,d$, or $b = 2\,d$.

Hence the shape is that of a half square.

In this case $R = \frac{b}{4}$.

(b) *Circular section.* — To find the level of water corresponding to maximum R (Fig. 48).

$$F = r^2\,(a - \tfrac{1}{2}\sin.\ 2\,a)$$

$$p = 2\,a\,.\,r$$

$$R = \frac{r}{4}\left(2 - \frac{\sin.\ 2\,a}{a}\right)$$

R will be a maximum when

$$2\,a = \tan.\ 2\,a\,.$$

Solving this by trial, we find

$$a = 128^\circ\ 43'\ 36.5''$$

Here we find

$$R = 0.629\,r\,.$$

(c) *Trapezoidal section.* —It is a theorem of geometry that of all equal polygons, with the same number of sides, the regular polygon has the minimum perimeter. Hence the shape of section for which R is a maximum will always be a half or an entire regular polygon; and for the trapezoidal section the semi-hexagon would give the greatest R.

Comparing polygons with different numbers of sides, it is a theorem of geometry that of all isoperimetric plane figures the circle has the largest area. Hence the greater the number of sides of a regular polygon, the greater will R be; and the absolute maximum of R will occur for the circle.

It is not always possible to make the trapezoidal section hexagonal in shape, with the sides sloping at 60°. Generally, a is given by the nature of the material, and, in this case, to find the most favorable form, we have to make $\frac{p}{F}$ a minimum, or

$$\frac{1}{T} = \frac{T\,(2 - \cos.\ a)}{F\sin.\ a}\,.$$

$$T = \sqrt{\frac{F\sin.\ a}{2 - \cos.\ a}}\,.$$

In this case

$$T = \frac{2F}{p} = \frac{B_1 - B_o}{2 \cot. a} .$$

$$p = \frac{2F}{T} = B_1 + B_o$$

$$R = \tfrac{1}{2} \sqrt{\frac{F \sin. a}{2 - \cos. a}} .$$

If it is desired to design a canal according to these principles, to carry a certain quantity of water, the steps to be taken are as follows: (1) Assume the velocity, with reference to the character of the bed. (2) Calculate $F = \frac{Q}{v}$. (3.) Assume a. (4.) Calculate the proper proportions. (5.) Find the necessary slope from the formulæ to be given. If this does not suit, a different v may be chosen, necessitating a change in F.

Distribution of velocity in cross-section. — The velocity varies in different parts of the same cross-section, and a knowledge of the relation between the mean velocity and the velocity at different points is of great importance, as we shall see. It is mean velocity that we have to deal with in calculating discharge, but it is, in most cases, impossible to measure this mean velocity.

Considering a vertical at any point of the stream, the velocity at different points is found to decrease from the bottom towards the surface. The best observations show that it reaches its maximum a short distance below the surface, this depth varying, according to the observations of Humphreys and Abbot on the Mississippi, from 0.1 to 0.5 of the total depth in the vertical considered, and being generally, in clear weather, at about 0.3 of the total depth. In very wide streams, however, like the La Plata near its mouth, the maximum velocity has been found at the surface.

Regarding this phenomenon, many explanations have been advanced. It was at first thought to be due to the retardation at the surface due to the friction of the atmosphere; but Humphreys and Abbot found that, although the direction of the wind influenced the position of the point of maximum velocity, yet even when the wind was down stream the maximum velocity occurred below the surface, thus showing that the friction of the atmosphere could not account for the phenomenon. The true cause is, probably, that, on account of resistances met with along the bed and banks, some of the slower-moving water near the bottom is transferred to the surface, coming up in boils in the current, and ascending almost continuously along the banks, and spreading out over the surface. The wider and deeper the stream, the more likely, therefore, the maximum velocity is to be at the surface.

If we lay off horizontally, from a vertical line representing the depth, the velocity at each point, we shall obtain a curve, called the vertical velocity curve. Many hypotheses have been made regarding its form, but the three principal ones are: (1) that it is a parabola, with horizontal axis a short distance below or at the surface (Humphreys and Abbot); (2) that it is a parabola with vertical axis, and vertex at the bottom or a short distance below, thus always giving the maximum velocity at the surface (Hagen); (3) that it is a straight line inclined to the vertical, thus always giving the maximum velocity at the surface (Revy and Weisbach). Call v the mean velocity in the vertical; v_o = surface velocity; v_s = velocity at bed; v_1 = maximum velocity, occurring at a depth t_1; v_x = velocity at a depth t_x; t = total depth in vertical: V = mean velocity in entire cross-section; then Humphreys' and Abbot's equation for the vertical velocity curve was

(a) $$(v_1 - v_x) = \sqrt{\frac{1.69\ V}{\sqrt{t} + 1.5}} \left(\frac{t_x - t_1}{t}\right)^2 \text{ or}$$

$$(t_x - t_1)^2 = P\ (v_1 - v_x)$$

Here we have

$$v = \tfrac{1}{3}\left[2\ v_1 + v_s + \frac{t_1}{t}\ (v_o - v_s)\right]$$

(b) Weisbach gave

$$v_x = \left(1 - 0.17\ \frac{t_x}{t}\right) v_o .$$

Here the mean velocity occurs at mid-depth, and equals $\frac{1}{2}\ (v_o + v_s)$

(c) Hagen assumed the equation of the velocity curve as follows:—

$$v_x = C + p\sqrt{h} .$$

where h = height above bottom.

Here the average velocity in the vertical is

$$v = C + \tfrac{2}{3}\ p\sqrt{t}$$

and occurs at a depth of $\frac{5}{9}\ t$.

Hagen recommends careful measurements at this depth, instead of many measurements at different points. But it would be more convenient still if we could find a relation between v and v_o, which would render unnecessary anything but a surface measurement. Hagen finds

$$v = v_o \frac{1 + 0.15\ \sqrt{0.9711\ t}}{1 + 0.225\ \sqrt{0.9711\ t}}$$

In a later work (1883) Hagen assumes the equation $y^5 = p\,h$, if y is the velocity at a height h above the bottom. Here

$$v = \frac{5}{6}\sqrt[5]{p\,t}$$

and $v = \frac{5}{6}\,v_o$, or entirely independent of the depth. Moreover, the mean velocity occurs, according to this law, at a height of 0.403 t above the bottom, or at a depth of 0.597 t.

In 1876, Hagen gave the equation

$$v = v_o\left(1 - 0.0322\sqrt{t}\right)$$

Humphreys and Abbot consider that their measurements showed that the ratio $\frac{v}{v_x}$ of the mean velocity to that at any point in the vertical is, approximately, constant for but one point, viz., at mid-depth, and that the mean velocity is equal to the velocity at mid-depth multiplied by a ratio a little less than one. Ellis, in his measurements on the Connecticut River, found the ratio to vary from 0.92 to 0.96, the average being 0.94. He also found the depth of the thread of mean velocity to be at from 0.622 to 0.656 of the total depth, the grand mean being 0.636 t.

Call v_o' = maximum surface velocity, v_s' = velocity at bed in same vertical with v_o'. Then Dubuat gave, from his experiments on artificial channels,

$$\mathrm{V} = \frac{v_o' + v_s'}{2}\,.$$

Prony gave the following: —

$$\frac{\mathrm{V}}{v_o'} = \frac{v_o' + 7.78}{v_o' + 10.34}$$

When v_o' was between 0.65 and 4.92 feet per second, he found that the equation $\mathrm{V} = 0.816\,v_o'$ was correct within 4 per cent, and $\mathrm{V} = 0.8\,v_o'$ was correct within 10 per cent.

None of these formulæ take any account of character of bed, or of the dimensions of the stream. Darcy and Bazin proposed the following, based on careful experiments: —

$$\frac{\mathrm{V}}{v_o'} = \frac{1}{1 + \sqrt{a + \frac{\beta}{\mathrm{R}}}}$$

The following are the values of a and β: —

	a.	β.
1°. Beds of smooth cement, without sand; or planed boards, carefully laid	0.00015	0.00001476
2°. Cement with sand; paved walls; brick; planks not planed	0.00019	0.000043624
3°. Rougher walls; stonework or paving . .	0.00024	0.0001968
4°. Earthen bed	0.00028	0.001148

These results are the best for artificial channels, because the experiments were made on such.

For rough calculations, we may assume:

$$\left.\begin{aligned} v &= 0.9\ v_o \\ \mathrm{V} &= 0.8\ v_o' \end{aligned}\right\}.$$

Formulæ for Flow.

General principles. —When water flows in open channels, its flow is governed by the laws of fluid friction. These laws are almost the reverse of those of the friction of solids, as will be seen from the following comparison: —

Friction of solids { varies as the pressure; is independent of surface; " " " velocity.

Friction of liquids { is independent of the pressure; varies with the surface; " " " velocity.

We consider here only the case of uniform and permanent motion, and in this case the channel is supposed to have everywhere the same section, and the water to stand at the same depth at every point. The bed must, therefore, be inclined, as the flow depends upon the inclination of the surface, which must, therefore, be parallel with the bed in order that the motion may be uniform, or the same at every cross-section.

Applying the theorem of Bernouilli to two points at a distance $d\ l$ apart, the only loss of head is that due to friction, which is

$$\theta = \frac{p\ d\ l\ .\ f\ (v)}{\mathrm{F}}$$

since friction varies as the area $p\ d\ l$, and as some function $f\ (v)$ of the velocity; the loss of head will evidently vary with $\frac{p\ d\ l}{\mathrm{F}}$. Hence, if we call v_o and z_o the velocity and elevation of the water surface at

some starting point, and v_n and z_n the corresponding quantities at some point at a distance l from the first, we shall have

$$\frac{v_o^2}{2g} + \frac{p}{\gamma} + z = \frac{v_n^2}{2g} + z_n + \frac{p_n}{\gamma} + \int_o^l \frac{p}{F}\, d\, l \,.\, f(v)$$

or since $p = p_n$ and $h = z - z_n$ is the difference in level or fall in the distance l,

$$\frac{v_n^2 - v_o^2}{2g} = h - \int_o^l \frac{p}{F}\, d\, l \,.\, f(v)$$

This is the general equation for permanent motion, either uniform (the same in all cross-sections) or varied. For uniform motion it becomes, since $v_o = v_n$:

$$h = \frac{p\, l}{F} \,.\, f(v)$$

or calling $\frac{h}{l} = i$, and $\frac{F}{p} = R$, $R\, i = f(v)$

This is the general equation for flow, and the basis of all the formulæ which have been proposed. These differ simply in the function of v according to which the different authors suppose the friction to vary.

Old formulæ. —Chézy assumed $f(v) = a\, v^2$, where a is a constant; hence,

$$v = c \sqrt{R\, i}$$

c being another constant, which is given by different writers, and for different kinds of channels, all the way from 60 to 100. Eytelwein's co-efficient, deduced from experiments by Dubuat, is 92.1 for feet measure.

Now v, in the above equation, is really the velocity *at the bed*, upon which the friction depends; but we intend it shall represent the mean velocity, because that is what we wish to determine; hence the co-efficient c must involve the ratio between these two velocities. But there is no constant ratio between them; hence c cannot possibly be a constant, and in fact it is found to vary considerably. All formulæ, however, may be reduced to this form, the only difference between them being in the value given to c, some authors making c vary as the cross-section changes, others with the velocity, others with the slope, etc.

Girard put $f(v) = a\, v + a\, v^2$.

Prony put $f(v) = A\, v + B\, v^2$, and gave the values

$$A = 0.00004445,$$
$$B = 0.0000943,$$

based on Dubuat's experiments.

Eytelwein determined the values of A and B, taking, in addition to Dubuat's experiments, others by Brünings, Woltmann, and Funk, and found

$$A = 0.0000243,$$
$$B = 0.000112.$$

Lahmeyer proposed $f(v) = a\, v_s^3$

St. Venant proposed $f(v) = a\, v_{11}^{21}$

Dupuit advocated $f(v) = a\, v_s + b\, v_s{}^2$, v_s being velocity at bottom.

Weisbach makes

$$f(v) = \frac{v^2}{2g}\left[0.007409\left(1 + \frac{0.19198}{v}\right)\right]$$

All these formulæ make $f(v)$ and c vary simply with the velocity; but experiments show it to vary with the slope, hydraulic mean depth, and, above all, with the character of the bed. Hence all the above formulæ are, at the present day, of no value.

NEWER FORMULÆ.

1°. *Humphreys* and *Abbot*, from their measurements on the Mississippi River, gave the formula

$$v = \left[\sqrt{0.0081\, b + \sqrt{\frac{225\ \mathrm{F}\ \sqrt{i}}{p + \mathrm{W}}}} - 0.09\ \sqrt{b}\right]^2$$

where w = width at water surface, and

$$b = \frac{1.69}{\sqrt{1.5 + \mathrm{R}}}$$

This formulæ may be applied to streams flowing in beds of *variable section, and with bends.* To apply it, proceed as follows: trace, approximately, the centre line of the current as a series of straight lines making deflection angles of 30° with each other; calculate $h' = \frac{v^2\ n\ \sin.\ 30^\circ}{134}$, where n = number of deflections, and v = assumed mean velocity of current; subtract h' from the fall h between the two end sections, and use the remainder in calculating i; in other words, assume the head h' to be lost in overcoming losses due to bends and changes of section. In using

the formula, all quantities should be average values for the stretch of river considered. This formula is applicable only to large streams, and *not* to smooth, artificial channels with uniform section. According to Gen. Abbot, it is applicable only when F is greater than 100 square feet, and i less than 0.0008. When F is smaller than this limit, Abbot proposes to subtract a term $\frac{2.4\sqrt{v'}}{1+p}$, v' being the velocity as found from the formula as originally given. It is better to limit the application of this formula to very large rivers, as we shall see that there are others giving better results for small streams.

2°. *Grebenau*, the translator of Humphreys' and Abbot's work into German, proposed to simplify the formula to

$$v = \beta \sqrt{225\, r \sqrt{i}} \quad ; \qquad \left(r = \frac{R}{2}\right)$$

as the terms omitted are generally small. Without the constant β the above formula gave too large a velocity as the stream was smaller. He gave values of β as follows: —

Small streams; less than 1 square meter in section . . .	0.8543
Streams; from 1 to 5 square meters in section	0.8796
Streams; from 5 to 10 square meters in section . . .	0.8890
Rivers; from 20 to 400 square meters in section . . .	0.9223
Large rivers; over 400 square meters in section . . .	0.9459

In this formula the constant c has the form $\frac{c'}{4\sqrt{i}}$.

Both these formulæ are not applicable to artificial channels, or streams with large fall.

3°. *Darcy* and *Bazin*. — These experimenters tried the four different kinds of bed enumerated on page 64, and falls from 0.001 to 0.009. They gave the formula

$$f(v) = \left(a + \frac{\beta}{R}\right) v^2$$

or, reduced to feet measure,

$$v = 1.81 \sqrt{\frac{1}{b_1}} \sqrt{R\, i}$$

in which b has the following values for the four categories of bed: —

I. $b_1 = 0.00015 \left(1 + \frac{0.0984}{\mathrm{R}}\right)$

II. $b_1 = 0.00019 \left(1 + \frac{0.2296}{\mathrm{R}}\right)$

III. $b_1 = 0.00024 \left(1 + \frac{0.81}{\mathrm{R}}\right)$

IV. $b_1 = 0.00028 \left(1 + \frac{4.1}{\mathrm{R}}\right)$

These are excellent formulæ, and applicable to all cases except where the fall is very small. For smooth, artificial channels of uniform section they are as good as any; but if the form of cross-section is much different from those experimented on (circular or trapezoidal) the results will be in error.

4°. *Gauckler's* formula, based on Darcy and Bazin's measurements, is as follows: —

$$\begin{cases} \text{for } i > 0.0007 : \sqrt{v} = a \sqrt[3]{\mathrm{R}} \sqrt[4]{i} \\ \qquad\qquad\qquad c = a^2 \sqrt[6]{\mathrm{R}} \\ \text{for } i < 0.0007 : \sqrt[4]{v} = \beta \sqrt[3]{\mathrm{R}} \sqrt[4]{i} \\ \qquad\qquad\qquad c = \beta^4 \sqrt[6]{\mathrm{R}^5} \sqrt{i} \end{cases}$$

Regarding this formula, it may be said that it is in principle wrong to have two formulæ, and that these do not agree well with experiments.

4°. *Bornemann's* formula, obtained by discussing Gauckler's, and adding some results of his own experiments, was

$$\sqrt[4]{v} = a \sqrt[3]{\mathrm{R}} \sqrt[5]{i}$$

This formula is also of little value.

6°. *Hagen's* formula. His first formula was $v = a \sqrt{\mathrm{R}} \sqrt[6]{i}$, in which $a = 4.38925$ for foot measure.

This formula, making no distinctions regarding character of bed, is, of course, valueless. Hagen's latest formulæ are

$$v = 4.90 \, \mathrm{R} \sqrt[5]{i} \text{ , for small streams.}$$

$$v = 6.045 \sqrt{\mathrm{R}} \sqrt[5]{i} \text{ , for large streams.}$$

These two values become equal for R = 1.52 feet; hence the former formula applies when R is less than this value, and the latter when it is greater. The bed is supposed of earth.

7°. *Ganguillet* and *Kutter's* formula is as follows: —

$$v = \frac{23 + \frac{1}{n} + \frac{0.00155}{i}}{0.55 + \left(23 + \frac{0.00155}{i}\right)\frac{n}{\sqrt{R}}} \sqrt{R\,i}$$

in which n is a co-efficient depending on the roughness of the bed, as follows: —

		n	$\frac{1}{n}$
1°.	Smooth cement, or carefully planed boards . .	0.010	100.00
2°.	Boards	0.012	83.33
3°.	Cut stone, or jointed brick	0.013	76.91
4°.	Rough stone	0.917	58.82
5°.	Earth; stream, and rivers	0.025	40.00
6°.	Streams carrying detritus, and with plants . .	0.030	33.33

This formula is the best yet proposed; it agrees with Darcy and Bazin's, and also with Humphreys and Abbot's measurements.

8°. *Harder* (1878) believes that his experiments near Hamburg show that both Darcy and Bazin's, and Ganguillet and Kutter's formulæ gave too small a velocity for small streams. He proposes,

1°. Very smooth bed:

$$v = \left(127.605 + 7.254\sqrt{R}\right)\sqrt{R\,i}.$$

2°. Smooth bed; boards; masonry; brick:

$$v = \left(1.0136 + 7.254\sqrt{R}\right)\sqrt{R\,i}.$$

3°. Earth, and rough masonry bed:

$$v = \left(65.65 + 7.254\sqrt{R}\right)\sqrt{R\,i}.$$

These formulæ agree quite well with experiments.

In applying these formulæ for flow the following problems may occur:

1°. Given, cross-section of channel, Q, and i; to find depth of water. (Solved by approximation, assuming depth at first.)

2°. Given, section of current, and i; to find Q. (Apply formulæ directly.)

3°. Given, section of current, and Q; to find i. (Calculate v, and solve formula for i.)

4°. Given, Q and i; to design cross-section. (Assume shape of section, and proceed as in 1°.)

Backwater. — If, in a channel in which water is flowing with permanent and uniform motion, the water-level be raised at some point, as by a dam, the motion becomes varied, and the depth of water behind the dam generally decreases up stream as far as the effect of the dam is felt; in other words, within that distance the water-level is raised above its former position, and the shape of the surface is not a plane, but a curve. It is often important to find the shape of this curve, and to determine how far the effect of such a raising of the water-level is felt. When the channel has a regular shape, mathematical solutions of this problem may be arrived at, and the equation of the water-surface found; but, in practice, the channel is never constant in shape, and an approximate solution is just as accurate, and more easy of understanding.

The first step is to make or procure a topographical map of the stream for some distance above the proposed site. Next, the height of dam being assumed, calculate how high the water will stand on its crest at the stage of the water for which it is desired to calculate the backwater. Starting with this water-level, the motion is considered uniform (the surface parallel to the bed) for a short distance. Applying Kutter's formula, we have everything given except i and n. The latter being assumed, calculate i. In assuming n, a gauging of the stream may be made, together with a determination of the slope, to find n more accurately. Having found i, the depth of water at the upper end of the section considered is found, a new R and v calculated, and so the operation is repeated, proceeding in short steps up stream.

Backwater due to partial obstructions, such as bridge-piers (Fig. 49). —Let A B be the natural surface of the water in a channel whose shape is known, and let it be obstructed by some bridge-piers of width w, and with a clear distance l between them. Then the water will be obliged to flow more rapidly between the piers, and hence the level of the water will be raised above them until the necessary additional head is produced, while between the piers the depth will be less than before. Then, referring to the figure:

Section above piers $= h$ L.

Section between p ers $= m\ h'\ l$,

m being the co-efficient of contraction at the head of the piers. Further, if Q is the discharge,

mean velocity above piers $= v = \frac{Q}{h\ L}$

mean velocity between piers $= v' = \frac{Q}{m\ h'\ l}$.

Hence, we have clearly

$$h - h' = 2 = \frac{Q^2}{2\ g}\left[\frac{1}{m^2\ h'^2\ l^2} - \frac{1}{h^2\ L^2}\right]$$

Generally, we may assume $h^1 = H$. Hence,

$$z = \frac{Q^2}{2\ g}\left[\frac{1}{m^2\ l^2\ H^2} - \frac{1}{L^2\ (H + z)^2}\right]$$

This equation is solved by successive approximation, and m will depend on the shape of the pier. Eytelwein found $m = 0.95$ when the front is triangular or oval, and $m = 0.85$ when it is square. Generally, it may be taken as 0.90.

CHAPTER IV.

Hydrometry.

Hydrometry treats of the measurement of quantities of water flowing in natural or artificial channels. The quantity may be found in four different ways:

1°. By direct measurement in a measuring vessel. This is only suitable for very small streams, and will not be further considered.

2°. By measurement of the flow through orifices or over weirs. This necessitates building a dam across the stream, and is only applicable to small streams, or to cases where facilities already exist. It consists simply in applying the formulæ already explained.

3°. By calculating from the measured slope and cross-section, according to one of the formulæ for flow. This is really a *calculation*, and not a *measurement* of flow, and does not belong under the present head.

4°. By measuring the velocity of the flowing water at different points in the cross-section, and thence determining the discharge. This is now to be considered.

INSTRUMENTS FOR MEASURING VELOCITY.—We must distinguish

1°. Velocity at the surface.

2°. Velocity at any point of the cross-section.

3°. Mean velocity in a vertical.

4°. Mean velocity in entire section.

All instruments are of two kinds:

(a) Floats.

(b) Stationary instruments.

1°. *Surface velocity.*—Measured by (a) floats; (b) log; (c) patent log; (d) hydrometric wheel; (e) any of the instruments used for velocity at any point.

2°. *Velocity at any point:*

(a) Double floats (Humphreys and Abbot; Ellis; Hagen; Cunningham; Gordon).

Advantages: convenient for great depths.

Disadvantages: error due to connecting cord; error due to surface float; wind; vertical movement of lower float.

Method of correcting error due to surface float, by having both floats of same size, and observing first surface velocity, and then velocity of floats connected. If $v_1 =$ surface velocity; $v_2 =$ velocity at lower float; $v_3 =$ velocity of connected floats; then

$$v_3 = \frac{v_1 + v_2}{2} .$$

$$v_2 = 2\, v_3 - v_1 .$$

(b) Castelli's hydrometric pendulum.

(c) Michelotti's hydraulic balance.

(d) Lorgna's hydraulic lever.

(e) Zimenes' hydraulic vane.

(f) Brünings' tacheometer.

These instruments are not now used. They are inconvenient for great depths, and inaccurate.

(g) Pitot's tube (1730), very inaccurate; height of column fluctuates, and cannot be read accurately; inconvenient for large depths.

$$v = c\sqrt{2gh}.$$

(h) Darcy's tube: modified form of Pitot's tube; quite accurate. Disadvantage at great depth, and that it only gives velocity at a particular moment, while it would be more accurate to get the average for a few moments.

Advantage: that it requires no measurement of time, and can be used close to bed and banks.

(i) Current meter, or Woltmann's wheel; first used by Woltmann before 1790.

This is the most generally applicable instrument. For great depths, it must be arranged to slide along a rope attached to a sinker.

Principal disadvantage of original instrument was that it had to be taken out of the water after each measurement to read the number of revolutions. This difficulty was obviated by Henry's electrical register, and Wagner's acoustic apparatus.

These instruments are rated experimentally by moving them with a known velocity through still water.

(j) Perrodil's torsion plate, for low velocities.

3°. *Mean velocity in a vertical:*

(a) Loaded tubes, or Cabeo's rod. First used by Cabeo in 1646. Now very generally used for smooth channels. Not generally applicable to channels of varying section and depth.

Tubes of different lengths are used for different depths of water. This instrument gives the average velocity in the depth taken by the tube, but as the latter can never reach to the very bottom, a correction must be applied to the results obtained by it. Francis gives this correction as follows: Let v_1 be the velocity found by the tube; then if d is the mean depth of the water along the course taken by the tube, and d' is the depth to which the tube is immersed, the true mean velocity in the vertical will be

$$v = v_1\left[1 - 0.116\left(\sqrt{D} - 0.1\right)\right]$$

in which $D = \frac{d - d'}{d}$.

This instrument is best applicable to rectangular flumes of constant section, and is so used in Lowell and Lawrence.

(b) The current meter may be so used as to give the mean velocity in a vertical, by starting it at the bottom, and moving it slowly and uniformly to the top. It is essential that this motion should not take place too rapidly, or at a rate not over five per cent of the velocity of the current. Such a measurement is called a vertical integration.

4°. *Mean velocity in entire cross-section.*—When the cross-section is rectangular, this may be determined by the current meter, by moving it diagonally from top to bottom, and back, moving it horizontally about one-fifth of the depth each time, making what is called a diagonal integration. When the channel is not rectangular, this method does not apply unless the distances it is moved horizontally each time be determined mathematically. The method is best adapted to rectangular flumes.

Of the instruments which have been named, the double float, current meter, Darcy's tube, and loaded tube, are in use most extensively at present.

Other methods of gauging small quantities of water have been proposed, depending upon analysis of the water, observation of the temperature, etc.

METHODS OF DETERMINING Q FROM MEASUREMENTS OF VELOCITY.

(a) Measurements with loaded tubes in rectangular flumes.

Plot the velocities on cross-section paper, distances from the side of the flume being abscissas. Divide the points obtained into consecutive groups, not more than ten observations to a group. Pass a regular curve among the points obtained in such a way that within each group the sum of the distances of the different points from the curve is zero. The area of this curve, multiplied by the depth, and corrected as explained, is the discharge.

(b) If v is the average velocity in any small area a, then the total discharge is $Q = \Sigma\, v\, a$. If the velocity has been measured at different points, this equation may be applied. Generally, the velocity is measured at a number of points in a series of equidistant verticals, or the mean velocity in these verticals is measured by integration. In either case, there are several methods of reducing the observations. (1) The mean velocity in each vertical may be considered the mean in the vertical strip of which the vertical is at the centre. (2) The average of the mean velocities in the verticals may be taken as approximately the mean velocity of the entire section.

Abbot gives the following methods of finding the mean velocity from measurements of the *mid-depth* velocities in a series of equidistant verticals: —

(1) The mean of all the mid-depth velocities [multiplied by 0.94] gives nearly the mean velocity in the section.

(2) The most exact method is as follows:—

In the formula

$$v_1 = \frac{1}{12}\sqrt{b\,v} \quad . \quad . \quad . \quad . \quad (1)$$

in which v_1 is the mid-depth velocity, v the mean velocity in the entire section, and

$$b = \frac{1.69}{1.5 + \sqrt{D}}$$

(D being the mean depth in a vertical strip), substitute for v_1 the mid-depth velocity in each strip, and multiply by the area of the strip; the sum of the equations so obtained will equal $v\,F$; or

$$v\,F = \Sigma\, v_1 f = \frac{1}{12}\,\Sigma\, f\sqrt{b\,v}$$

f being the area of a strip.

The lesser root of this equation is the mean velocity in the cross-section.

(3) $$v = \left[\left(1.08\,u_1 + 0.002\,b\right)^{\frac{1}{2}} - 0.045\,\sqrt{b}\,\right]^2$$

In this equation u_1 = mean of the mid-depth velocities, and

$$b = \frac{1.69}{\sqrt{1.5 + R}}$$

CHAPTER V.

Theory of the Planimeter.

In fig. 50, B D is the area to be measured; B is the tracing-point of the instrument, attached to the arm A B, hinged at A to the arm A O, O being the fixed point of the instrument. The point A thus describes arcs of a circle, with O as a centre and O A = r as radius, while B may describe figures of different shapes. Let A B and A′ B′ be two consecutive

positions of the arm A B, and call A B $= a$, and A C $= b$. At C a roller is placed, with axis parallel to A B, so that it revolves at every motion at right angles to A B. Call the angle A O A$' = d\theta$, and the angle that A A$'$ makes with the horizontal $= \theta$. Produce A B and A$'$ B$'$ till they meet in M, and call the angle B M B$' = d\phi$ and the angle that A B makes with the horizontal ϕ. Then we have the rotation of the roller equal to M C . $d\phi = du$.

To find this, we must first find M C:

$$\text{A M} : \text{A A}' :: \text{sin. A A}' \text{M} : \text{sin. A M A}' ;$$

$$\text{or, A M} : r\,d\theta :: \text{sin. } (\phi - \theta) : d\phi$$

$$\text{A M} = \frac{r\,d\theta - \text{sin. } (\phi - \theta)}{d\phi} \quad \therefore$$

$$du = r\,d\theta \,.\, \text{sin. } (\phi - \theta) + b\,d\phi$$

$$u = r \int \text{sin. } (\phi - \theta)\, d\theta + b \int d\phi \,.$$

But as the point B, after traversing the circumference of a figure, returns to the starting-point, $\int d\phi = o$; hence,

$$u = r \int \text{sin. } (\phi - \theta)\, a\,\theta \,.$$

We have now to find an expression for the area of the figure B D. The area A B B$'$ A$' =$ the parallelogram A B B$''$ A$' +$ the triangle A$'$ B$''$ B$'$, or

$$\text{A B B}' \text{A}' = a \,.\, r\,d\theta \,.\, \text{sin. } (\phi - \theta) + \tfrac{1}{2} a^2\, d\phi \,.$$

$$\therefore \quad \text{A} = a\,r \int \text{sin. } (\phi - \theta)\, d\theta$$

since the integral of the last term becomes zero. This expression, then, represents the area B D. Comparing it with the value of u, we see that

$$\text{A} = a\,u$$

Let $c =$ circumference of roller,

$n =$ number of revolutions made.

Then $\text{A} = a\,u = a\,n\,c$

If $a\,c = 1$; then $\text{A} = n$.

Planimeters have also been made which give the statical moment and the moment of inertia of any plane figure about any axis.

The above demonstration supposes that the point O is outside of the figure whose area is to be measured, so that the integral of $d\,\phi$ is zero. But another case arises if the point Q is within the area considered, for in this case the integral of $d\,\phi$ will be $2\,\pi$, and we shall have

$$u = r \int \text{sin. } (\phi - \theta)\, d\,\theta + 2\,\pi\, b.$$

$$A' = a\, r \int \text{sin. } (\phi - \theta)\, d\,\theta + \pi\, a^2$$

But in this case the area A′ does not represent the area to be measured, but that comprised between the outline traced and the circle described by the point A, whose radius is r; hence

$$A = a\, r \int \text{sin. } (\phi - \theta)\, d\,\theta + \pi\, a^2 + \pi\, r^2$$

$$= a\, u - 2\,\pi\, a\, b + \pi\, a^2 + \pi\, r^2$$

The last three terms $\pi\, a^2 + \pi\, r^2 - 2\,\pi\, a\, b$ represent the area of a *circle of correction*, whose radius is $\sqrt{a^2 + r^2 - 2\, a\, b}$, and whose area is always to be added to the result given by the instrument in this case. It is determined by measuring a large circle of known area.

CHAPTER VI.

The Flow of Water in Pipes.

The flow of water in pipes differs from that in open channels, in that in the former case the pressure may vary at different points along the pipe, while in the case already treated, the pressure at all parts of the channel on the water-surface is the same. From this it follows that, whereas in open channels no velocity can be produced without a slope of the water-surface, in a pipe, water may be made to flow up-hill, any increase of elevation and of velocity being compensated by a diminution of pressure. The laws of the motion may be deduced by applying the theorem of Bernouilli, taking account of the losses, which may be due: (a) to friction of the liquid particles among themselves and upon the sides of the pipe; (b) to bends and curves; (c) to branches, and (d) to sudden enlargements, which produce a sudden diminution of velocity.

The velocity in a closed pipe under pressure varies in different points of the cross-section, being least at the circumference, and greatest at the centre. The law of change of velocity has been investigated mathematically, and it has been determined that the mean velocity occurs at a distance from the centre equal to from two-thirds to seven-tenths of the radius of the pipe. In our equation for flow in pipes, we wish to consider the mean velocity in the entire section; hence, since the friction upon the sides is due not to that mean velocity, but to the velocity at the circumference, the same remarks apply here that were made on page 65 regarding the constant c in the formulæ for flow in open channels.

The formulæ for flow in pipes depend simply upon the determination of the losses due to the causes just named. For we may apply the theorem of Bernouilli as follows to the case shown in fig. 51: Let the elevation of the centre of the tube at A and B be respectively z and z'; let the pressures at those points be respectively p and p', and let the velocities be v and v'. Then we have clearly

$$z + \frac{p}{\gamma} + \frac{v^2}{2g} = z' + \frac{p'}{\gamma} + \frac{v'^2}{2g} + \text{losses};$$

the losses to be considered being those which arise between A and B. Let P be the point to which the liquid would rise in a closed tube inserted at A, and Q the corresponding point for B. Then the difference of level of P and Q is $z + \frac{p}{\gamma} - \left(z' + \frac{p'}{\gamma}\right)$, which we may call h. Hence,

$$h = \left(\frac{v'^2}{2g} - \frac{v^2}{2g}\right) + \text{losses}.$$

We proceed to find the values for the losses:

(a) *Loss due to friction.*

Call d the diameter; r the radius; p the perimeter; F the area, of a pipe of a uniform diameter; v the velocity; Q the quantity passing in one second; and l the length considered. Then the loss due to friction in the distance l will be

$$h' = \frac{p\ l}{\mathrm{F}} \cdot f(v) = \frac{4\ l}{d} f(v) \quad :$$

or, if $\frac{h'}{l} = i$;

$$\tfrac{1}{4}\ d\ i = f(v)$$

If the pipe is of uniform section, or v and d are constant, then i is constant. In this case, the only loss is that due to friction, and it is rep-

resented by h in fig. 51, so that i is the sine of the angle which the line P Q makes with a horizontal.

It will be convenient to define here two terms which are of frequent use. The *hydraulic gradient* of a pipe is the line obtained by connecting all the points obtained by laying off at each point of the pipe the quantity $z + \frac{p}{\gamma} + \frac{v^2}{2g}$. Thus at the point A, in fig. 51, the hydraulic gradient would be at a distance $\frac{v^2}{2g}$ above P, while at B it would be $\frac{v'^2}{2g}$ above Q. If there were no losses this line would be straight and horizontal. If there were the same loss in every horizontal foot of the pipe, as would sensibly be the case if the pipe were of uniform section, and with no branches, curves, or obstructions, it would be a straight but inclined line. In reality it is a broken line, with sudden drops at places where sudden losses of head occur. In practice, it is convenient to consider the hydraulic gradient not as above defined, but at a distance $\frac{p_o}{\gamma}$ below the line described, p_o being the atmospheric pressure; this pressure being therefore neglected, and simply the excess of pressure above the atmospheric considered. We shall in future consider the line in this sense.

If one point of the hydraulic gradient is given, we may construct the line by drawing through the given point a horizontal, and laying off below or above that horizontal, according as we proceed in the direction in which the water is flowing, or in the contrary direction, the losses of head occurring between the given point and the point in question. Thus, in fig. 52, which represents a pipe-line connecting two reservoirs, the upper line represents the hydraulic gradient.

The *pressure-line* is the locus of the points representing $z + \frac{p}{\gamma}$, such as P and Q, in fig. 51, except that we consider p as simply the excess over the atmospheric pressure. Hence the pressure-line lies at a distance $\frac{v^2}{2g}$ below the hydraulic gradient, and represents the height to which water would rise in a tube open to the atmosphere, instead at any point of the pipe. If we can draw, in any case, the hydraulic gradient, and the pressure-line, we can solve all problems which may occur.

The different formulæ for flow differ only in regard to the value of $f(v)$ assumed in the term representing loss of head due to friction. This value is found from experiments. Thus, in fig. 52, if h_1, h_2, etc., represent losses due to other causes than friction, we shall have

$$h = \frac{v^2}{2g} + \frac{p}{\gamma} + \frac{l}{\mathrm{F}} \cdot f(v) + h_1 + h_2 + \ldots .$$

$$f(v) = \frac{h - h_1 - h_2 - \ldots - \frac{v^2}{2g}}{l} \cdot \frac{\mathrm{F}}{p} .$$

and, by measuring Q, d, and h, and arranging to have no losses h_1, h_2, etc., we may find $f(v)$.

We have found the loss due to friction, which may be expressed

$$h' = \frac{l}{d} \cdot f(v)$$

We shall also express this loss in the following form: —

$$h' = \theta \frac{l}{d} \cdot \frac{v^2}{2g},$$

a form which will be found very convenient in calculation.

(a) *Prony* (1802).

$$h' = \frac{l}{d}\left(0.0000693256\, v + 0.000424\, v^2\right)$$

$$h' = \left(0.0273571 + \frac{0.0044646}{v}\right) \frac{l}{d} \cdot \frac{v^2}{2g} .$$

(b) *d'Aubuisson.*

$$h' = \frac{l}{d}\left(0.0000752\, v + 0.000416\, v^2\right)$$

$$h' = \left(0.02679 + \frac{0.00484}{v}\right) \frac{l}{d} \cdot \frac{v^2}{2g} .$$

(c) *Eytelwein* (1814).

$$h' = \frac{l}{d}\left(0.0000888\, v + 0.000341376\, v^2\right)$$

$$h' = \left(0.02196 + \frac{0.00571872}{v}\right) \frac{l}{d} \cdot \frac{v^2}{2g} .$$

(d) *Dupuit.*

$$h' = \frac{l}{d} \cdot \left(0.00047\, v^2\right)$$

$$h' = 0.030268 \frac{l}{d} \cdot \frac{v^2}{2g} .$$

(e) *St. Venant* (1851).

$$h' = \frac{l}{d}\left(0.000506\, v^{\frac{12}{7}}\right)$$

$$h' = \frac{0.0325864}{v_1^2} \cdot \frac{l}{d} \cdot \frac{v^2}{2g} .$$

(f) *Weisbach.*

$$h' = \frac{l}{d}\left(0.000266\ v_2^3 + 0.000223\ v^2\right)$$

$$h' = \left(0.01439 + \frac{0.0171427}{\sqrt{v}}\right)\frac{l}{d}\cdot\frac{v^2}{2g}$$

All these formulæ were founded on some old experiments of Couplet, Bossut, and Dubuat, Weisbach adding eleven of his own.

(g) *Darcy* began in 1849, and completed in 1851, a series of experiments the most valuable that have ever been made. He experimented on pipes of iron, lead, glass, etc., new and old, with diameters of from 0.5 inch to 20 inches, and with velocities from 6 inches to 17 feet per second. He showed that in new pipes the friction varies considerably with the nature and polish of the surface, that this effect is gradually lost as the pipe becomes covered with deposits, and he assumed

$$f(v) = a\ v + b\ v^2,$$

adding, that if the pipes have been in use some time, it will be sufficient to put

$$f(v) = b_1\ v^2.$$

As all pipes are liable to deposits, it is safest to use the last formula; hence,

$$h' = \frac{l}{d}\left(0.0006184 + \frac{0.00005176}{d}\right)v^2$$

$$h' = \left(0.039825 + \frac{0.003333}{d}\right)\frac{l}{d}\cdot\frac{v^2}{2g}.$$

This formula is for pipes partially coated with deposits. Darcy considered that these deposits would double the friction, so he doubled the co-efficients obtained from clean pipes. For clean pipes we should have

$$h' = \left(0.01991 + \frac{0.001666}{d}\right)\frac{l}{d}\cdot\frac{v^2}{2g}$$

The general value of the quantity in the parentheses for the range of sizes in ordinary use will not vary much from the following:

For clean pipes, ...	0.0207
For old pipes, ...	0.0414

(h) *Hagen* proposed, in 1869, a formula based upon Darcy's experiments, taking account of the temperature, which he considered to have considerable effect upon the co-efficients.

Other formulæ have been given by other experimenters, but of all those which have been given the best to use are Darcy's and Weisbach's. The former is best applicable in all ordinary cases occurring in practice, in questions of water supply. The latter agrees best with the experiments where the velocity is very large,— say over 20 feet per second.

We have considered simply the loss due to friction. The losses due to bends and angles have been determined experimentally by Weisbach, but in most practical problems the values of such losses are so small compared with that due to friction that they may be neglected. The same may be said of the loss due to branches.

Regarding sudden enlargements, we have seen that if the velocity v is suddenly diminished to v_1 there is a loss equal to $\frac{(v-v_1)^2}{2g}$. Thus, in fig. 52, at the entrance to the pipe, if such entrance is not rounded, the loss will be $\frac{1}{2}\cdot\frac{v^2}{2g}$; while, at the lower reservoir, the loss will be $\frac{v^2}{2g}$. Valves cause losses depending upon the same general principle of a contraction of the water-way, and the value of the losses may be given; but, in many questions, such losses may be neglected.

Application of the equations.

Pipe of uniform section. —Draw on the profile of the pipe-line a horizontal line through a given point A on the hydraulic gradient. If the pipe runs from a reservoir, this line may be the level line through the surface or the water in the reservoir. Let h be the distance from this level line down to the *pressure line* at a point distant l from A. Then we shall have the two following equations, which suffice for the solution of all cases in practice: —

$$Q = \frac{\pi d^2}{4}\cdot v \quad \ldots \quad (1)$$

$$h = \frac{v^2}{2g} + \text{losses}$$

$$= \frac{v^2}{2g} + \theta_1\frac{v^2}{2g} + \theta_2\frac{v^2}{2g} + \ldots + \theta'\frac{l}{d}\cdot\frac{v^2}{2g} \quad \ldots \quad (2)$$

In eq (2), all the terms, except the first and last, represent losses of head due to valves, curves, branches, etc.; θ_1, θ_2, etc., expressing the fractions of the height due to the velocity. The last term is the loss due to friction. It is to be remarked here, that, in practice, the hydraulic

gradient and the pressure-line may be considered as identical, inasmuch as they are only $\frac{v^2}{2g}$ apart, which can rarely be much over a foot, in ordinary cases.

In these two equations there are four quantities which may be unknown, namely, Q, d, v, and h. Any two of these being given, the other two may be found, and the following problems thus arise: —

(1) Given d and h, to find v and Q; or to find the quantity and velocity in a pipe of given diameter, with a given loss of head.

The problem is solved by finding the value of v from (2), and substituting it in (1). Thus, from (2),

$$v = \sqrt{\frac{2\,g\,h}{1 + \theta_1 + \theta_2 + \ldots . \theta' \frac{l}{d}}} \quad \ldots . \quad (3)$$

and, substituting in (1), we find

$$Q = \frac{\pi\,d^2}{4} \sqrt{\frac{2\,g\,h}{1 + \theta_1 + \theta_2 + \ldots . \theta' \frac{l}{d}}}$$

If we may neglect all losses except friction, we have more simply

$$v = \sqrt{\frac{2\,g\,h}{\theta'} \cdot \frac{d}{l}} \quad . \quad . \quad . \quad . \quad (4)$$

$$Q = \frac{\pi\,d^2}{4} \sqrt{\frac{2\,g\,h}{\theta'} \cdot \frac{d}{l}} = 0.7854 \sqrt{\frac{2\,g\,d^5}{\theta'} \cdot \frac{h}{l}} \quad . \quad . \quad . \quad (5)$$

If we call θ' equal to 0.0414, we have

$$Q = 30.98 \sqrt{\frac{d^5\,h}{l}} \quad . \quad . \quad . \quad . \quad (6)$$

(2) Given d and v; to find h and Q. h is found directly from eq. (2), and Q from eq. (1).

(3) Given d and Q; to find h and v. v is found directly from eq. (1), and then h from eq. (2).

(4) Given v and Q; to find d and h. d is found from eq. (1), and h from eq. (2) at once.

(5) Given v and h; to find d and Q. d is found by successive approximation from (2), and then Q from (1).

(6) Given h and Q; to find d and v. Find the values by successive approximation. Or, if we may neglect all losses but friction, we have from (5)

$$d = \sqrt[5]{\frac{8\ Q^2\ \theta'\ l}{\pi^2\ g\ h}} \quad 0.48 \quad = \sqrt[5]{\frac{\theta' .\ Q^2\ l}{h}}$$

or, calling $\theta' = 0.04144$,

$$d = 0.25 \sqrt[5]{\frac{Q^2\ l}{h}} .$$

This last equation is the one most used in practice, and tables may be calculated giving values of d for various values of Q and h.

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NOTES ON HYDRAULICS.

PREPARED FOR THE USE OF THE STUDENTS

OF THE

CIVIL ENGINEERING DEPARTMENT

OF THE

MASS. INSTITUTE OF TECHNOLOGY,

BOSTON, MASS.

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BOSTON:
W. J. SCHOFIELD, PRINTER, 105 SUMMER STREET.
1885.

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